Geodesic Flows

- **Hopf** (1939): ergodicity of geodesic flows on manifolds of *constant* negative curvature
- **Anosov** (1963): ergodicity of geodesic flows in negative curvature
- **Collet-Epstein-Gallavotti** (1984), **Moore** (1987), **Ratner** (1987), **Pollicott** (1992): exponential decay of correlations for geodesic flows in *constant* negative curvature (two and three dimensions)
- **Sinai–Ruelle–Bowen** (early '70), **Ruelle** (1978), **Pollicott** (1985): thermodynamics formalism and dynamical ζ function
- **Chernov** (1998): sub-exponential decay of correlations for geodesic flows on negative curvature surfaces
- **Dolgopyat** (1998): exponential decay of correlations for Anosov flows with $\mathcal{C}^1$ foliations (e.g.: negative curvature surfaces and 1/4-pinched manifolds)

- **Ruelle** (1983): Example of a suspension over a shift with piecewise constant ceiling which is mixing but with arbitrarily slow correlation decay.

Theorem 1 *For each $\mathcal{C}^4(\mathcal{M}, \mathcal{M})$ Anosov Contact flow T_t there exists $\sigma > 0$ such that, for each $\alpha \in (0, 1]$, there exists $c_\alpha > 0$ such that: for each $f, \varphi \in \mathcal{C}^\alpha$, $t > 0$*

$$\left| \int f \varphi \circ T_t - \int f \int \varphi \right| \leq c_\alpha |f|_{\mathcal{C}^\alpha} |\varphi|_{\mathcal{C}^\alpha} e^{-\alpha\sigma t}.$$

By *Anosov* we mean that at each point $x \in \mathcal{M}$ the tangent space to the manifold $\mathcal{M}$ splits into three subspaces $\mathcal{T}_x\mathcal{M} = E^s(x) \oplus X(x) \oplus E^u(x)$, where X is the flow direction, with the following properties: there exists $A, \sigma > 0$,

- $dT_t E^{s/u}(x) = E^{s/u}(T_t x)$ (*invariance*)
- for each $v \in E^s(x)$, $t \geq 0$, $\|dT_t v\| \leq Ae^{-\sigma t}\|v\|$ (*stable contracts*)
- for each $v \in E^u(x)$, $t \geq 0$, $\|dT_t v\| \geq Ae^{\sigma t}\|v\|$ (*unstable expands*)

In the following we will assume $A = 1$.

It is a remarkable fact of hyperbolic theory that the above infinitesimal properties imply the existence of stable and unstable manifolds at every point

Here, by *Contact* I mean that $\mathcal{M}$ is an odd $2d+1$ manifold and there exists a smooth one form ω, invariant under the flow, such that $\omega \wedge (\wedge^d d\omega)$ is a volume form.

The existence of such a form is well known for geodesic flows.

Let $\mathtt{M}$ be a manifold with negative curvature and $\mathcal{M}$ its unitary tangent bundle. Let $T_t : \mathcal{M} \to \mathcal{M}$ be the geodesic flow.

Let, for $f \in L^1(\mathcal{M}, \mathbb{C})$,

$$\mathcal{L}_t f := f \circ T_{-t}.$$

Then, setting $d\mu := f\, dm$, holds

$$T_t^* \mu(\varphi) := \mu(\varphi \circ T_t) = \int_{\mathcal{M}} f \varphi \circ T_t = \int_{\mathcal{M}} \mathcal{L}_t f \varphi,$$

thus the knowledge of the composition operators $\mathcal{L}_t$ (*but on which space?*) can help in studying the converge to equilibrium.

Consider the following norms

$$\|f\|_w := \sup_{\substack{\varphi \in \mathcal{C}^0 \\ |\varphi|_{s,1} \le 1}} \int_{\mathcal{M}} f\varphi$$

$$\|f\| := \|f\|_u + \|f\|_s$$

$$\|f\|_u := H_{u,1}(f)$$

$$\|f\|_s := \sup_{\substack{\varphi \in \mathcal{C}^0 \\ |\varphi|_{s,\beta} \le 1}} \int_{\mathcal{M}} f\varphi$$

where $|\varphi|_{s,\alpha} := |\varphi|_\infty + H_{s,\alpha}(\varphi)$; and

$$H_{s,\alpha}(\varphi) := \sup_{x \in \mathcal{M}} \sup_{y \in W^s_\delta(x)} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\alpha},$$

$H_{u,\alpha}$ being defined analogously. From now on $\mathcal{L}_t$ will act on the Banach spaces $\mathcal{B}(\mathcal{M},\mathbb{C})$, $\mathcal{B}_w(\mathcal{M},\mathbb{C})$ obtained by closing $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ with respect to $\|\cdot\|$ and $\|\cdot\|_w$ respectively.

$\mathcal{L}_t$ IS A "REGULARIZING" SEMI GROUP

Lemma 1 (Lasota-Yorke type inequality) *$\{\mathcal{L}_t\}_{t\geq 0}$ is a strongly continuous semi group on $\mathcal{B}$ and $\mathcal{B}_w$. In addition,*

$$\|\mathcal{L}_t f\|_w \leq \|f\|_w \; ; \quad \|\mathcal{L}_t f\| \leq \|f\|$$

$$\|\mathcal{L}_t f\| \leq 3e^{-\lambda\beta t}\|f\| + B\|f\|_w.$$

Hence there exists a generator $X : D(X) \subset \mathcal{B} \to \mathcal{B}$ and the following formula for the resolvent $R(z) := (z - X)^{-1}$ holds true

$$R(z) = \int_0^\infty e^{-zt}\mathcal{L}_t dt \; ; \quad \forall z \in \mathbb{C}, \Re(z) > 0.$$

Let us have a glance at the proof.

$$
\begin{aligned}
H_{u,1}(\mathcal{L}_t f) &= \sup_{x\in\mathcal{M}} \sup_{y\in W^u_\delta(x)} \frac{|f(T_{-t}x) - f(T_{-t}y)|}{d(T_{-t}x, T_{-t}y)} \frac{d(T_{-t}x, T_{-t}y)}{d(x,y)} \\
&\leq e^{-\sigma t} H_{u,1}(f)
\end{aligned}
$$

Analogously, $H_{s,\alpha}(\varphi\circ T_t) \leq e^{-\alpha\sigma t} H_{s,\alpha}(\varphi)$, thus $|\varphi\circ T_t|_{s,1} \leq |\varphi|_{s,1}$ and

$$
\|\mathcal{L}_t f\|_w = \sup_{\substack{\varphi\in\mathcal{C}^0 \\ |\varphi|_{s,1}\leq 1}} \int_{\mathcal{M}} f\varphi\circ T_t \leq \|f\|_w.
$$

The rest is more or less as silly as this.

$R(z)$ is even better

Lemma 2 *Let $z = a + ib$, $a > 0$, then*

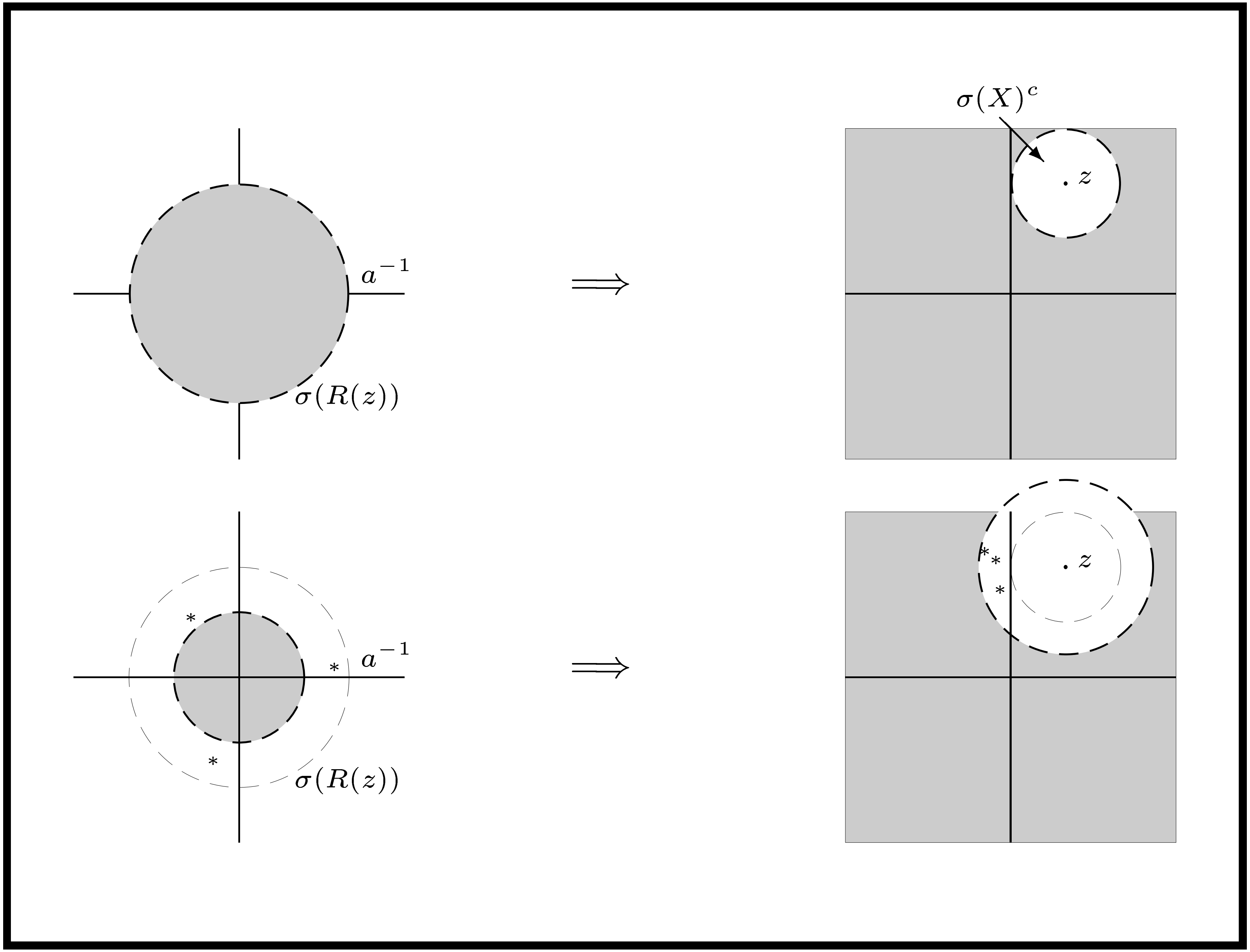
$$\|R(z)\|_w \leq a^{-1}\ ; \quad \|R(z)\| \leq a^{-1}$$

$$\|R(z)^n f\| \leq \frac{3}{(a+\lambda\beta)^n}\|f\| + a^{-n}B\|f\|_w.$$

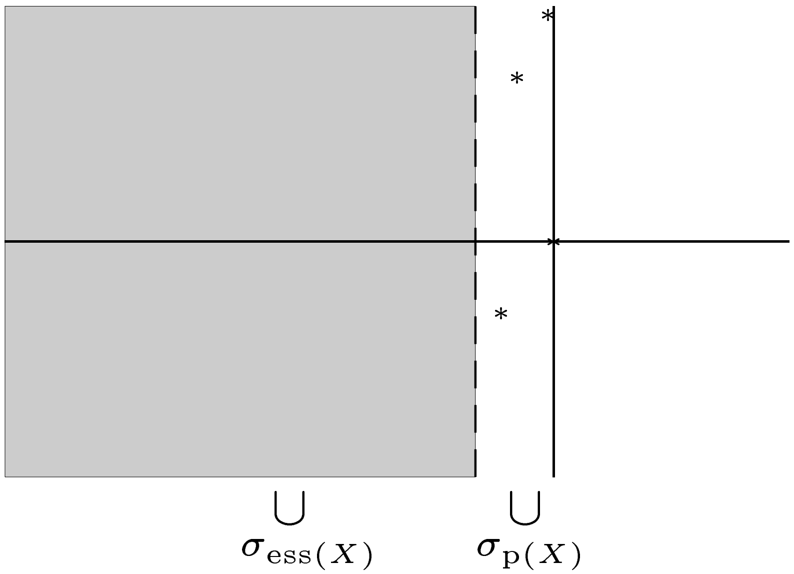
In addition, $R(z) : \mathcal{B}(\mathcal{M}, \mathbb{C}) \to \mathcal{B}_w(\mathcal{M}, \mathbb{C})$ is compact.

We can then investigate the spectrum of X by studying the resolvent.

Notice: $\zeta \in \rho(R(z))$ is equivalent to $z - \zeta^{-1} \in \rho(X)$.



a^{-1}
$\sigma(R(z))$
$\Longrightarrow$
$\sigma(X)^c$
z
a^{-1}
$\sigma(R(z))$
$\Longrightarrow$
z



The ergodicity of the flow implies that 0 is a simple eigenvalue, while the mixing implies $\sigma(X) \cap \{ib\}_{b\in\mathbb{R}} = \{0\}$.

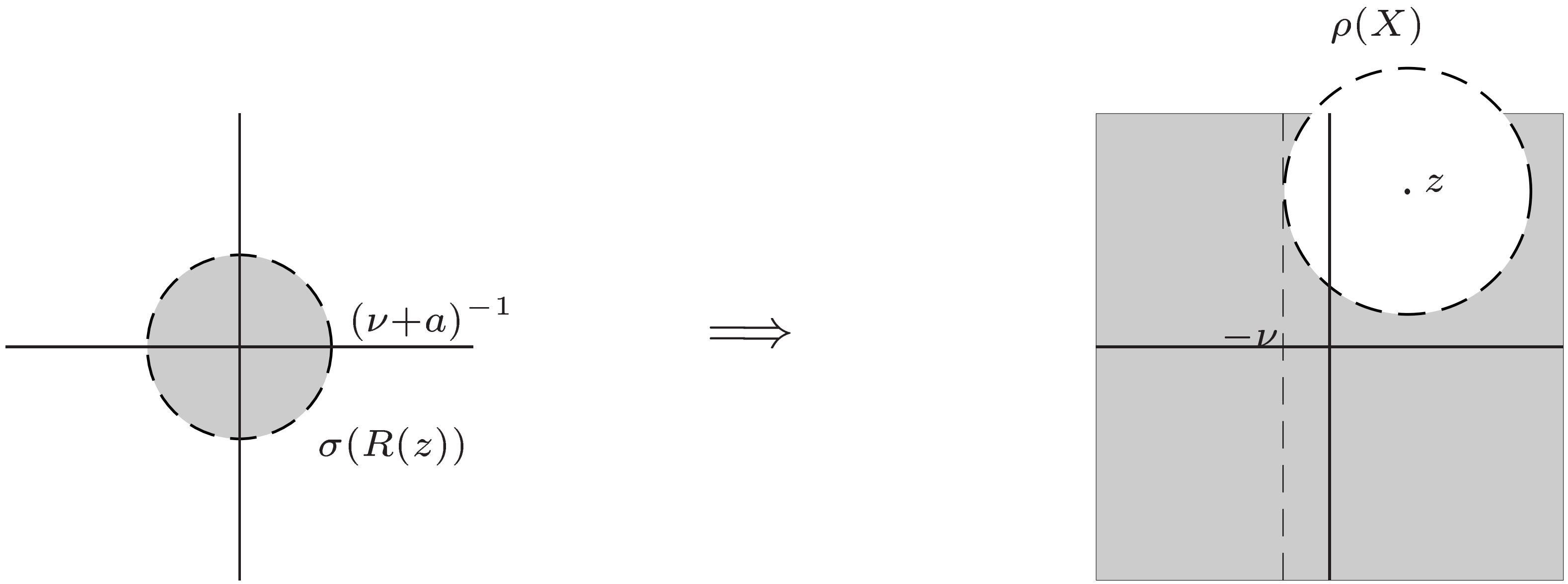
Up to now the results hold for each Anosov flow and they correspond to the existence of a strip on which the Laplace transform of the correlation functions (or the dynamical zeta function) is meromorphyc.

Lemma 3 (Dolgopyat inequality) *There exist* $c, b_*, \nu > 0$ *such that, for all* $a \in [1,2]$, $|b| \geq b_*$, $z = a + ib$, *and* $n \geq c \ln |b|$,

$$\|R(z)^n\| \leq (a+\nu)^{-n}.$$

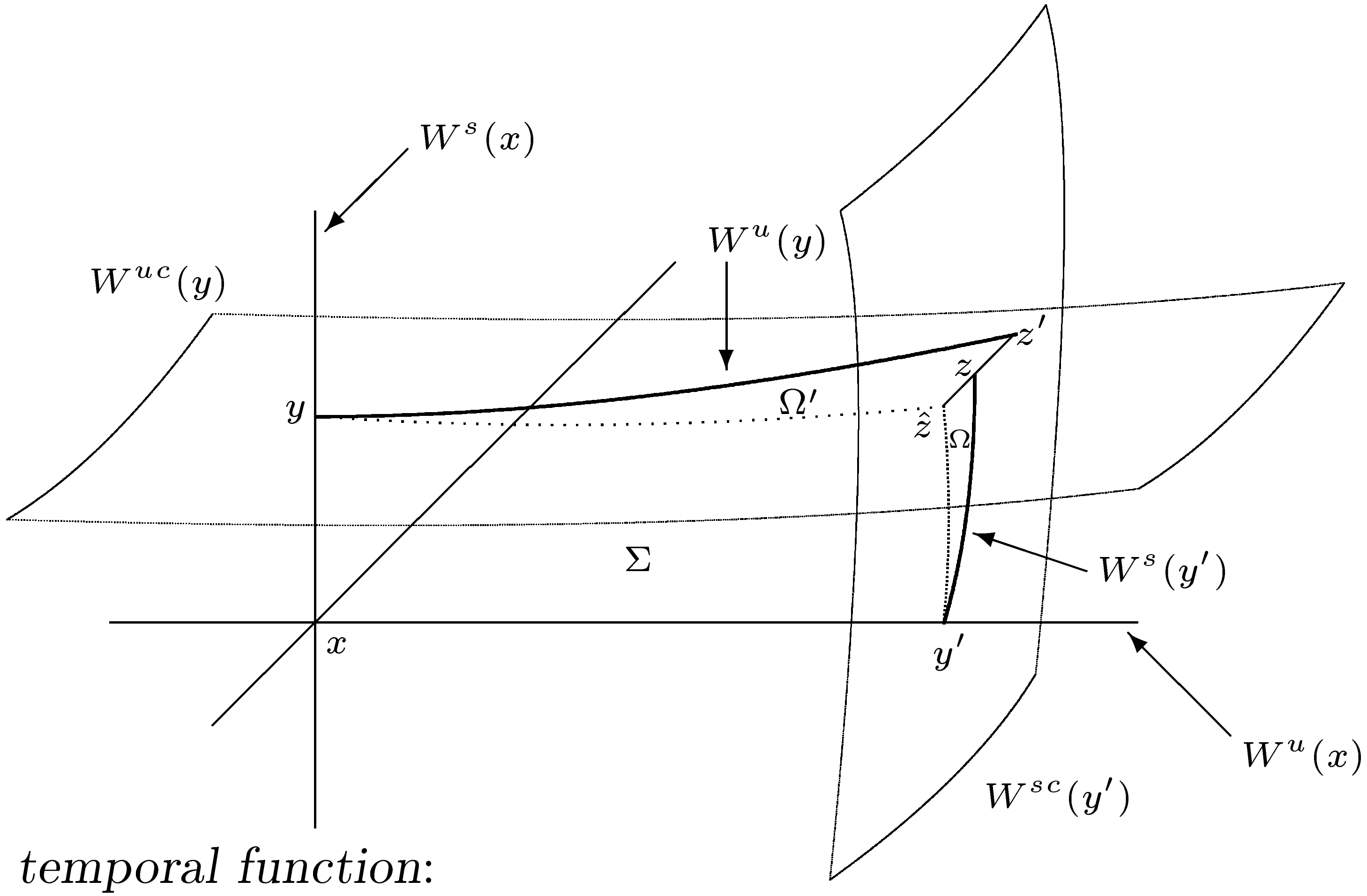
Hence,

$$r_{\mathrm{sp}}(R(z)) \leq (a+\nu)^{-1}.$$



Dolgopyat result holds for a different operator on a different space, yet the cancellation mechanism relevant to the proof is similar. In his case he uses $\mathcal{C}^1$ foliations. Here the contact structure suffices to yield the result regardless the regularity of the foliation.

In both cases the crucial point is a precise control of the **temporal function** (Chernov) expressing the non joint integrability of the foliations.



The *temporal function*:

Let $\exp_x(v) = y'$ and $\exp_x(w) = y$, then

$$d(z, z') =: \Delta(y, y') = d\alpha(v, w) + \mathcal{O}(|v|^\tau |w|^2 + |w|^\tau |v|^2).$$

Finally the Theorem is proven by applying the inverse Laplace transform.

Lemma 4 *If $f \in D(X^2)$, then*

$$\mathcal{L}_t f = \frac{1}{2\pi} \lim_{w \to \infty} \int_{-w}^{w} e^{at+ibt} R(a+ib) f \, db.$$

Lemma 4 yields, for $f \in \mathcal{C}^1$ with $\int f = 0$,

$$\|\mathcal{L}_t f\| \leq C_1 e^{-\sigma t} |f|_{\mathcal{C}^1}.$$

and the result follows by a standard approximation argument.

Let us say few words about the proof.

$$H_{u,1}(R(z)f) \leq \frac{1}{a+\lambda} H_{u,1}(f); \quad H_{s,\beta}(R(z)^*\varphi) \leq \frac{1}{a+\lambda\beta} H_{s,\beta}(\varphi)$$

are the basis to prove Lemmata 1, 2. From Lemma 2 follows

$$\begin{aligned}
\|R(z)^{3n}f\| &\leq \frac{3}{(a+\lambda\beta)^n}\|R(z)^{2n}f\| + a^{-n}B\|R(z)^{2n}f\|_w \\
&\leq 3(a+\lambda\beta)^n a^{2n}\|f\| + a^{-n}B\|R(z)^{2n}f\|_w \\
&\leq \frac{1}{2(a+\nu)^{3n}}\|f\| + a^{-n}B\|R(z)^{2n}f\|_w
\end{aligned}$$

To prove Lemma 3 we need only estimate the weak norm of $R(z)^n f$ in terms of the strong norm of f.

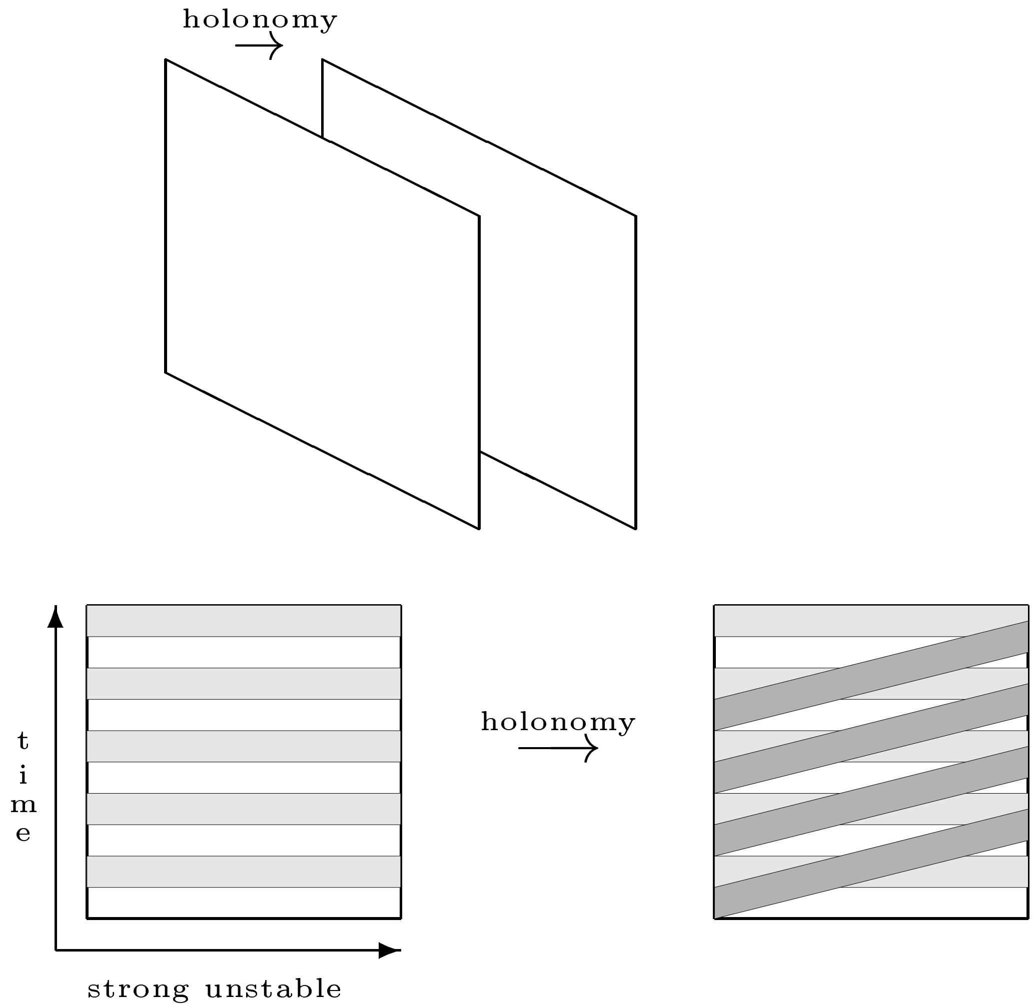
$$\int R(z)^{2n} f\varphi \sim \int f R(z)^{*n} \mathbb{A}^u_\delta R(z)^{*n} \varphi$$

Where $\mathbb{A}^u_\delta$ is the average of a piece of unstable manifold of size δ.

$$\mathbb{A}^u_\delta R(z)^{*n}\varphi(x) = \frac{1}{(n-1)!} \int_{\cup_{t=0}^{\infty} T_t W^u_\delta(x)} t^{n-1} e^{-zt} \varphi(\xi) J_u T_{-t}(\xi)$$

Note that, for large t, the image of the unstable manifold becomes extremely long and invades all the space.

The idea is then to sum together different pieces of such a long manifold to get cancellations.


holonomy
time
strong unstable
holonomy