ERGODIC PROPERTIES FOR A QUANTUM NON LINEAR DYNAMICS

FRANCESCO FIDALEO AND CARLANGELO LIVERANI

Abstract. We present a quantum system composed of infinitely many particles, subject to a non quadratic Hamiltonian, for which it is possible to investigate the long time behavior of the dynamics and its ergodic properties. We do so both for the KMS states and for a large class of locally normal invariant states, whose very existence is already of some interest.

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1. Introduction

The last years have witnessed a growing interest in the ergodic properties of non relativistic quantum systems. For example, on the one hand a large literature has been devoted to the so called "quantum chaos," that is to the study of the spectrum of the Hamiltonians whose classical associated motion enjoys strong ergodic properties (see [12, 16, 28] for an overview). On the other hand, there has been renewed interested in the study of the convergence to equilibrium in quantum systems (i.e. in the study of strong statistical properties directly in the quantum setting) [11, 13, 20, 24, 25, 33]. Of course, in order for such ergodic properties to be present at all, the quantum system must have infinitely many degree of freedom, otherwise only quasi periodic motion can take place.

The present paper illustrates, and investigates in details, a simple, but non trivial, class of models of infinite non relativistic quantum systems that exhibit relaxation to equilibrium. Such models are examples of one dimensional harmonic crystals with defects and are akin to the one studied in [8] from a different point of view. Indeed, the model consists of an infinite harmonic chain of particles with a special particle

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subject to a non harmonic external potential. It can also be thought as a model for a particle subject to an external potential and in contact with a thermal bath (see [8] and references within).

The aim of our analysis is not only to obtain a full understanding of the examples at hand, which already present interesting features, but also to use these concrete examples as a guide into the subtleties of the matter. In particular, on the one hand our work illustrates the rôle played by the choice of the right set of observables, on the other hand it suggests that some regularity requirements, usually assumed in the study of C^*-dynamical systems (see e.g. [7, 24]), may be too stringent to be applicable to the cases under consideration, which fall into a very general setting.

In contrast with much work present in the literature, our models are not exactly solvable (the dynamics it is not linear) and we go beyond KMS states.

We first show that, when the dynamics is linear, the above systems are mixing (or strongly clustering, according to terminology in current literature) with respect to a large class of quasi-free states. This result is neither surprising nor completely new. Namely, similar results exist for classical and quantum systems (see e.g. [11, 19, 29]) and only a technical difficulty–in our systems the spectrum of the generator of the classical dynamics may not be bounded away from zero–prevents us from presenting our results as a corollary of previous ones. Remark that the restriction to quasi-free states is rather natural since, under some technical conditions, all invariant states for a linear dynamics are quasi-free [5].

Next, we consider the case in which a non-harmonic potential, on the special particle, is present (in fact, we consider bounded analytic perturbations of a harmonic potential). It is well known that, in such a situation, the dynamics can be expressed via a perturbation expansion. The novelty here consists in the possibility to control the series uniformly in time when the perturbation it is not too large. Thanks to this, we are able to investigate the asymptotic properties of our model. We prove that the KMS states are mixing once the algebra of observables has been properly restricted. More surprisingly, to each quasi-free state, invariant for the linear evolution, it is associated an invariant state for the perturbed evolution and such states are mixing, provided the original one is.

This last result seems to point to a persistence of the integral of motions under perturbations and may suggest that the stability theorems which characterize the KMS state as the only stable state [7, 18, 26] employ a definition of "stability" too strong to be physically relevant– at least in some cases. We think this possibility deserves to be further investigated, eventually considering more general perturbations and different integrable models as a starting point.

The above mentioned smallness assumption on the non quadratic part of the external potential is such that we cannot change the convexity of the potential. Accordingly, we cannot consider a double well potential. It is not clear to us if this limitation is just of a technical nature or if it corresponds to a physical obstruction. For example, it could be that for a double well potential localization appears in the ground state (see [30] for a similar phenomenon in a different model). This problem requires further study.

The mathematical machinery employed is borrowed from operator algebras. As we always deal with dynamical systems where the time evolution has continuity property weaker than norm continuity, we cannot work directly in the framework of C^*-dynamical systems. Moreover, the ergodic properties that we want to investigate are not likely to hold on "natural" von Neumann algebras. Thus also W^*-dynamical systems are not adequate to our needs. In view of such a situation we introduce the following setting.

We consider the couple $(\mathfrak{A}, \alpha_t)$ where $\mathfrak{A}$ is a C^*-algebra and $t \to \alpha_t$ is a representation of the additive group of real numbers into the group of the automorphisms of the C^*-algebra $\mathfrak{A}$, that is a dynamical system.

We say that a representation π of $\mathfrak{A}$ on a separable Hilbert space gives rise to a *continuous dynamical system* if the maps

$$t \in \mathbb{R} \to \pi(\alpha_t A)$$

are all continuous in the weak (equiv. strong, strong*) operator topology.

As it is widely known, there are very simple examples satisfying the above picture, e.g. the dynamical system associated to a free particle. Moreover, if we deal with a group of Bogoliubov transformations $t \to T_t$ of a CCR C^*-algebra built on a sympletic space (H, σ), the Gelfand-Naimark-Segal (GNS for short) representation π_ω relative to any state ω gives rise to a continuous dynamical system according to the above definition, provided that both functions

$$\begin{aligned} t \in \mathbb{R} \to& \sigma(u, T_t v) \\ t \in \mathbb{R} \to& \omega(W(u + T_t v)) \end{aligned}$$

are continuous for fixed $u, v \in H$.

It is straightforward to show that, if ω is an invariant state satisfying the above conditions, then the time evolution is implemented on the GNS Hilbert space $\mathcal{H}_\omega$ by an one-parameter weakly continuous unitary group. Thus, the GNS triple relative to ω leads to a continuous covariant representation for the system $(\mathfrak{A}, \alpha_t, \omega)$.

In this paper, a triple $(\mathfrak{A}, \alpha_t, \omega)$ made of the dynamical system $(\mathfrak{A}, \alpha_t)$ together with a state ω such that the GNS representation π_ω gives rise to a continuous dynamical systems in the above sense, is said to be a *Quantum Dynamical System* (QDS for short).

Concerning ergodic properties of a QDS $(\mathfrak{A}, \alpha_t, \omega)$ with ω invariant, we introduce the following definitions (see e.g. [14, 18]):

(i) $(\mathfrak{A}, \alpha_t)$ is *asymptotically abelian* if

$$\lim_{t\to\infty} \|[\alpha_t A, B]\| = 0$$

for every $A, B \in \mathfrak{A}$.

(ii) $(\mathfrak{A}, \alpha_t, \omega)$ is *ergodic* if ω is the only invariant state for α_t in $\pi_\omega(\mathfrak{A})''_*$.

(iii) $(\mathfrak{A}, \alpha_t, \omega)$ is *weakly mixing* if

$$m_t(\omega(A(\alpha_t C)B) - \omega(AB)\omega(C)) = 0 \tag{1.1}$$

for every $A, B, C \in \mathfrak{A}$ and m_t is any invariant mean on $\mathbb{R}$.

(iv) $(\mathfrak{A}, \alpha_t, \omega)$ is *mixing* if

$$\lim_{t\to\infty} \omega(A(\alpha_t C)B) = \omega(AB)\omega(C) \tag{1.2}$$

for every $A, B, C \in \mathfrak{A}$.

We note that a state ω enjoying properties (iii) or (iv) is referred, in the current literature, as a *weakly clustering* or *strongly clustering* state respectively, see e.g. [6, 7, 14, 18, 24]. Yet, in analogy with the classical case, we prefer to retain our terminology since properties (iii) and (iv) are the natural generalizations to quantum (i.e. non commutative) cases of analogous properties considered in commutative cases, see e.g. [4, 15].

Noticing that we have the chain of implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) for the above properties ([18], Theorem 2.1), we focus our attention only on properties (i) and (iv).

Finally, we remark that, for asymptotically abelian QDS $(\mathfrak{A}, \alpha_t, \omega)$, property (iv) is equivalent to

$$\lim_{t\to\infty} \omega(A\alpha_t B) = \omega(A)\omega(B) \tag{1.3}$$

for every $A, B \in \mathfrak{A}$ (two-point clustering).[1] We will always use the above remark when proving mixing.

A tool often used in C^* or W^*-dynamical systems in order to investigate asymptotic properties is the following strengthening of (i):

(v) $(\mathfrak{A}, \alpha_t)$ is L^1-*asymptotically abelian* if there exists a norm-dense $*$-subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ such that

$$\int_0^\infty \|[\alpha_t A, B]\| dt < +\infty$$

for every $A, B \in \mathfrak{A}_0$.

We will employ a similar, but weaker, condition in our analysis, although we do not find convenient to state it in such general terms (basically, we require (v) for a much smaller set of operators).

The paper is composed by three parts.

The first one (Sections 2–4) contains the presentation of the class of models under consideration together with some of their basic properties. The linear case and ergodic properties of some of its quasi-free states are also investigated in great detail.

The dynamics, as well as ergodic properties of the non-linear case (obtained perturbing the linear system via a non-quadratic potential), is intensively analyzed in the second part (Sections 5, 6) of the paper.

The last part includes three appendices where, for the reader's convenience, we have collected all the needed computations.

2. The Model and the results

We begin our analysis considering the Hamiltonian

$$\begin{aligned} H(q,p) =& \frac{1}{2m}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\left(\frac{1}{M} - \frac{1}{m}\right) p_0^2 + \frac{K_0}{2}\sum_{i\in\mathbb{Z}} (q_{i+1} - q_i)^2 \\ &+ \sum_{i\in\mathbb{Z}} \frac{\kappa}{2} q_i^2 + V_*(q_0)\,. \end{aligned}$$

[1]For KMS states of C^*-dynamical or W^*-dynamical systems, the equivalence between (1.2) and (1.3) holds independently of asymptotic abelianess.

For simplicity we choose the unity of measure such that $m = 1$, $4K_0 = 1$ and $\hbar = 1$. Thus we obtain

$$\begin{aligned} H(q,p) =& \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\left(\frac{1}{M}-1\right)p_0^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 \\ &+ \sum_{i\in\mathbb{Z}}\frac{\kappa}{2}q_i^2 + V_*(q_0)\,. \end{aligned} \tag{2.1}$$

As a preliminary step, in the first part of the paper we will consider the special case in which $V_*(q) = \frac{K}{2}q^2$, $K > 0$, which yields the quadratic Hamiltonian

$$\begin{aligned} H(q,p) =& \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\left(\frac{1}{M}-1\right)p_0^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 \\ &+ \sum_{i\in\mathbb{Z}}\frac{\kappa}{2}q_i^2 + \frac{K}{2}q_0^2\,. \end{aligned} \tag{2.2}$$

Clearly H represents an infinite harmonic chain with a particle of different mass subject to an external potential. We will consider both the case $\kappa > 0$ and the case $\kappa = 0$, when an infrared divergence is present.

In order to study the evolution of such a (quantum) system, let us consider the real vector space $L^2_{\mathbb{R}}(\mathbb{Z})^2 := L^2_{\mathbb{R}}(\mathbb{Z}) \oplus L^2_{\mathbb{R}}(\mathbb{Z})$ of the doubly infinite sequences that are square summable and the symplectic space $(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)$ with symplectic form

$$\sigma(v,\,w) = \frac{1}{2}\sum_{i\in\mathbb{Z}} v_i^1 w_i^2 - v_i^2 w_i^1\,,$$

for each $v = (v^1,\,v^2)$, $w = (w^1,\,w^2) \in L^2_{\mathbb{R}}(\mathbb{Z})^2$.

We introduce the block operator $A_\mu : L^2_{\mathbb{R}}(\mathbb{Z})^2 \to L^2_{\mathbb{R}}(\mathbb{Z})^2$

$$A_\mu = \begin{pmatrix} 0 & I - a\mathcal{P} \\ K\Delta + \mu I - K\mathcal{P} & 0 \end{pmatrix}$$

where $\mu = -\kappa$,[2] $a = 1 - \frac{1}{M}$ and $\mathcal{P}$, Δ are operators from $L^2_{\mathbb{R}}(\mathbb{Z})$ to $L^2_{\mathbb{R}}(\mathbb{Z})$ defined by $(\mathcal{P}v)_i = \delta_{i0}v_0$ and $(\Delta v)_i = v_{i+1} + v_{i-1} - 2v_i$, respectively.[3]

[2]This notation is convenient since the parameter μ can be interpreted as a kind of "chemical potential" in analogy with [7], Section 5.2.

[3]The Hamilton equations associated to (2.2) read

$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = A_\mu \begin{pmatrix} q \\ p \end{pmatrix}\,.$$

It is well known (see e.g.[1, 2, 3, 7, 17, 21, 22, 34]) that, in a quantum system, the observables are described by a CCR algebra over (appropriate subspace of) $L^2_{\mathbb{R}}(\mathbb{Z})^2$. More precisely, to each symplectic subspace $H \subset L^2_{\mathbb{R}}(\mathbb{Z})^2$, it is uniquely associated a C^*-algebra $\overline{\mathfrak{A}(H,\sigma)}$.[4]

The algebras of observables that will be relevant in the sequel are determined by the Banach spaces $W^n(\mathbb{Z})$ made of sequences $v \equiv \{v_k\}$ such that

$$\|v\|_{W^n} := \left[\sum_{k\in\mathbb{Z}}(|k|^n + 1)|v_k|\right] < +\infty$$

(of course $W^0 = L^1$).

It is immediate to verify that A^*_μ generate a one-parameter group of Bogoliubov automorphisms (simply the adjoint of the classical evolution)

$$T^\mu_t v := e^{tA^*_\mu}v, \quad t \in \mathbb{R} \tag{2.3}$$

on each of the sympletic spaces $(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)$ and $(W^n_{\mathbb{R}}(\mathbb{Z})^2, \sigma)$. Thus, the dynamics relative to the Hamiltonian (2.2) is uniquely determined on all C^*-algebras $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ and $\overline{\mathfrak{A}(W^n_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ by the following action on the generators:

$$\alpha^{\mu,0}_t W(v) := W(T^\mu_t v), \quad t \in \mathbb{R}. \tag{2.4}$$

In Section 4.1 we will see that the C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ is asymptotically abelian w.r.t. the dynamics $\alpha^{\mu,0}_t$, $\mu < 0$, whereas the C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ is asymptotically abelian w.r.t. the dynamics $\alpha^{0,0}_t$ (Proposition 4.1).

Next, in Section 4.2, we introduce a wide class of quasi-free states ω for the case $\mu < 0$. We show that the associated Quantum Dynamical System $(\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}, \alpha^{\mu,0}_t, \omega)$ is mixing (Proposition 4.3). Such states do include the KMS states and ground states (see Section 4.4).

The case with $\mu = 0$ is analyzed in Section 4.3 where a class of quasi-free states ω, containing KMS states, is studied as well. Also in this case, the QDS $(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}, \alpha^{0,0}_t, \omega)$ is mixing (Proposition 4.6).

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[4] Given a symplectic subspace $H \subset L^2_{\mathbb{R}}(\mathbb{Z})^2$, the associated CCR C^*-algebra $\overline{\mathfrak{A}(H,\sigma)}$ is generated by the Weyl operators $\{W(v)\}_{v\in H}$ with the relations $W(0) = I$, $W(-v) = W(v)^*$, and $W(v)W(w) = e^{-i\sigma(v,w)}W(v+w)$, [17]. Notice that the C^* norm is uniquely determined by the algebraic structure [7], Theorem 5.2.8.

The definition of the dynamics $\alpha_t^{\mu,P}$ associated to the Hamiltonian (2.1) with

$$V_*(x) := \frac{K}{2}x^2 + V(x)$$

is more delicate. The main obstacle is that the operator associated to the potential V is not necessarily an element of the above C^*-algebras.

To be more precise we will consider

$$V(x) = \int_{\mathbb{R}} e^{i\lambda x}\nu(d\lambda)$$

where ν is a bounded complex measure with real Fourier transform. Clearly, the self-adjoint operator P naturally associated to $V(x)$ should be

$$P := \int_{\mathbb{R}} W(\lambda e_{0,1})\nu(d\lambda)$$

where $e_{0,1} = (e_0, 0)$ and, as usual, $e_0 = (\dots, 0, 1, 0, \dots)$. Unfortunately the above integral has meaning only in any of weak topologies associated to a suitable representation π of $\mathfrak{A} = \overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ (e.g. the weak*-topology of $\pi(\mathfrak{A})''$), not in the norm one. It is thus necessary to enlarge in a canonical way our C^*-algebras in order to define QDSs in the non linear case. This new algebra of observables will depend on the state ω under consideration. Note that the idea to choose the von Neumann algebra associated to the representation would not be too god: on such an enormous algebra there is no reason why the system should enjoy any interesting ergodic property at all. The above program is carried out in section 6, where suitable C^*-algebras $\mathfrak{M}_\omega$ are introduced.

Our main result can then be summarized by the following theorem.

Theorem 2.1. *Suppose that the potential V is sufficiently small. Then the following assertions hold.*

(i) *To any state ω on $\mathfrak{A}$ invariant w.r.t. the linear dynamics $\alpha_t^{\mu,0}$, such that $v \in W^2_{\mathbb{R}}(\mathbb{Z})^2 \to \omega(W(v)) \in \mathbb{C}$ is a Borel function, it corresponds a unique state ω_∞ on $\mathfrak{M}_\omega$ invariant w.r.t. the non linear dynamics $\alpha_t^{\mu,P}$, which can be obtained as the following pointwise limit:*

$$\omega_\infty(A) = \lim_{t\to\pm\infty} \omega(\alpha_t^{\mu,P}A)\,.$$

(ii) *If the QDS $(\mathfrak{A}, \alpha_t^{\mu,0}, \omega)$ is mixing, then the QDS $(\mathfrak{M}_\omega, \alpha_t^{\mu,P}, \omega_\infty)$ is mixing.*

The meaning of "sufficiently small" in the above theorem is specified by (6.1), in particular V must be analytic.

3. Preliminaries about the linear model

Even though much of the analysis of the linear model can be carried out by general considerations, to study ergodic properties of our specific models it is necessary to have detailed information on the spectral properties of the infinitesimal generator of the time evolution.

In order to obtain a more explicit representation of the time evolution (2.3) relative to the quadratic Hamiltonian (2.2), we use the discrete Fourier transform.

If we define $F : L^2(\mathbb{Z}) \to L^2(-1,1)$ by

$$Fv := \sum_{k\in\mathbb{Z}} v_k e^{i\pi kx},$$

then it is easy to show that $F(L^2_{\mathbb{R}}(\mathbb{Z}))$ is precisely made of all functions $\{f \in L^2(-1,1) \,|\, f = Cf\} =: L$ where the conjugation C is defined by

$$(Cf)(x) := \overline{f(-x)}\,. \tag{3.1}$$

As F is a unitary map, it is an isometry on $\tilde{L} := F(L^2_{\mathbb{R}}(\mathbb{Z})^2)$ equipped with the natural scalar product

$$\langle f,\, g\rangle = \frac{1}{2}\int_{-1}^{1} \bar{f}g\,.$$

Moreover, F realizes a sympletic map between $L^2_{\mathbb{R}}(\mathbb{Z})^2$ and $\tilde{L}$, provided we equip the last space with the symplectic form

$$\sigma(f,g) = \frac{1}{2}\left(\langle f^1,\, g^2\rangle - \langle f^2,\, g^1\rangle\right), \tag{3.2}$$

where $f = (f^1, f^2)$ and $g = (g^1,\, g^2)$ are element of $\tilde{L}$ (by a slight abuse of notation we have used σ to designate the symplectic form both on $\tilde{L}$ and $L^2_{\mathbb{R}}(\mathbb{Z})^2$).

A straightforward computation yields

$$\Lambda_\mu := FA_\mu F^{-1} = \begin{pmatrix} 0 & (I - a\tilde{\mathcal{P}}) \\ \Omega + \mu I - K\tilde{\mathcal{P}} & 0 \end{pmatrix}$$

where $\tilde{\mathcal{P}} : L \to L$ and $\Omega : L \to L$ are defined by $\tilde{\mathcal{P}}f = \frac{1}{2}\int_{-1}^{1} f$ and $(\Omega f)(x) = \frac{1}{2}(\cos(\pi x) - 1)f(x) := \omega(x)f(x)$. Note that, setting $c = M-1$, $(I - a\tilde{\mathcal{P}})^{-1} = I + c\tilde{\mathcal{P}}$ and $\Omega + \mu I - K\tilde{\mathcal{P}} = \Omega + \mu(I + c\tilde{\mathcal{P}}) - b\tilde{\mathcal{P}}$, with $b = K + \mu c$.

We are interested in studying analytic functions of A_μ, hence of Λ_μ. In particular, we need an explicit expression for $e^{\Lambda_\mu^* t}$.

Setting
$$\Gamma := I - a\tilde{\mathcal{P}}, \quad D := \Omega - b\tilde{\mathcal{P}},$$
we have
$$\Theta_\mu := (I - a\tilde{\mathcal{P}})(\Omega + \Gamma^{-1}\mu - b\tilde{\mathcal{P}}) \equiv \Gamma D + \mu I, \tag{3.3}$$
in addition,
$$\Theta_\mu \Gamma = \Gamma \Theta_\mu^*$$
(Γ and D are both self-adjoint).

The importance of (3.3) is clear if we compute the dynamics (2.3) via its Taylor expansion. We obtain
$$\begin{aligned} e^{\Lambda_\mu^* t} = & \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \begin{pmatrix} \Gamma^{-1}\Theta_\mu^n \Gamma & 0 \\ 0 & \Theta_\mu^n \end{pmatrix} \\ & + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & \Gamma^{-1}\Theta_\mu^{n+1} \\ \Theta_\mu^n \Gamma & 0 \end{pmatrix} \end{aligned} \tag{3.4}$$
where the series converge in norm, uniformly on bounded sets of $\mathbb{R}$.

In conclusion, we need to develop a functional calculus for the operator
$$\Theta := \Theta_0 = \Gamma D = \Omega - \tilde{\mathcal{P}}(\rho I + a\Omega)$$
where $\rho = b(1-a)$. Since the spectrum of Ω is $[-1, 0]$, and it is all essential spectrum, the spectrum of Θ (which is a perturbation of Ω by a rank one operator) must contain $[-1, 0]$, see [9]. In addition, for all $z \notin [-1, 0]$ for which
$$\delta(z) = -\{1 - \langle 1,\, (\rho I + a\Omega)(\Omega - zI)^{-1} 1\rangle\}^{-1} \tag{3.5}$$
it is well defined, it is easy to verify that
$$R_\Theta(z) := [I - \delta(z)(\Omega - zI)^{-1}\tilde{\mathcal{P}}(\rho I + a\Omega)](\Omega - zI)^{-1} \tag{3.6}$$
is the inverse of $\Theta - zI$ (just multiply on the left and on the right), that is the resolvent of Θ. Consequently the study of the spectrum of Θ is reduced to the study of $\delta(z)$.

From Appendix A we have, for all $z \notin [-1,0]$,

$$\delta(z) = -\left\{1 - a + \frac{2(\rho + az)}{\sqrt{(1+2z)^2 - 1}}\right\}^{-1} \tag{3.7}$$

which is well defined for $z \notin [-1,0]$ only if $1 + \frac{K}{1+\kappa} < M < 1 + \frac{K}{\kappa}$; otherwise it has one pole on $\mathbb{R}\setminus[-1,0]$ which corresponds to an eigenvalue of the operator Θ (for details and the exact definition of the square root see Appendix A). We also note that, as (see (3.1) for the definition of C)

$$\Theta C = C\Theta,$$

it is easy to verify that

$$CR_\Theta(z)C = R_\Theta(\bar{z}), \tag{3.8}$$

$$\Gamma^{-\frac{1}{2}} R_\Theta(z) \Gamma^{\frac{1}{2}} = \Gamma^{\frac{1}{2}} R_{\Theta^*}(z) \Gamma^{-\frac{1}{2}}. \tag{3.9}$$

Summarizing we have

Proposition 3.1. *If f is a function analytic in a neighborhood of $[-1,0]$, $1 + \frac{K}{1+\kappa} < M < 1 + \frac{K}{\kappa}$ and γ_c is a Jordan curve in the domain of analyticity of f and surrounding counterclockwise $[-1,0]$. Then*

(i) $f(\Theta) = -\frac{1}{2\pi i}\int_{\gamma_c} f(z) R_\Theta(z) dz$,
where the integral is understood as a Bochner integral.

(ii) *The operator $\Gamma^{-\frac{1}{2}} f(\Theta) \Gamma^{\frac{1}{2}}$ is a normal operator.*

(iii) *If $f|_{[-1,0]}$ is real, then*
(a) $Cf(\Theta)C = f(\Theta)$,
(b) $\Gamma^{-\frac{1}{2}} f(\Theta)\Gamma^{\frac{1}{2}}$ *is self-adjoint.*

Proof. (i) Since we show in Appendix A that, under the above hypothesis on M, the spectrum of Θ is $[-1,0]$, the first part of the proposition follows from the standard analytic functional calculus [9].

(ii) We choose a sufficiently small rectangle surrounding $[-1,0]$, symmetric w.r.t. the usual complex conjugation, as the contour γ_c. Thus,

$$\begin{aligned}\left(\Gamma^{-\frac{1}{2}} f(\Theta)\Gamma^{\frac{1}{2}}\right)^* &\equiv \left(-\frac{1}{2\pi i}\int_{\gamma_c} f(z)\Gamma^{-\frac{1}{2}} R_\Theta(z)\Gamma^{\frac{1}{2}} dz\right)^* \\ &= -\frac{1}{2\pi i}\int_{\gamma_c} f^*(z)\Gamma^{-\frac{1}{2}} R_\Theta(z)\Gamma^{\frac{1}{2}} dz \\ &\equiv \Gamma^{-\frac{1}{2}} f^*(\Theta)\Gamma^{\frac{1}{2}}\end{aligned}$$

since $f^*(z) := \overline{f(\bar{z})}$ is also analytic in a neighborhood of $[-1,0]$. The second identity follows by (3.9) after an elementary change of variable in the integral. Then

$$
\begin{aligned}
&\left(\Gamma^{-\frac{1}{2}}f(\Theta)\Gamma^{\frac{1}{2}}\right)^* \Gamma^{-\frac{1}{2}}f(\Theta)\Gamma^{\frac{1}{2}} = \Gamma^{-\frac{1}{2}}f^*(\Theta)f(\Theta)\Gamma^{\frac{1}{2}}\\
&= \Gamma^{-\frac{1}{2}}f(\Theta)f^*(\Theta)\Gamma^{\frac{1}{2}} = \Gamma^{-\frac{1}{2}}f(\Theta)\Gamma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}f^*(\Theta)\Gamma^{\frac{1}{2}}\\
&\equiv \Gamma^{-\frac{1}{2}}f(\Theta)\Gamma^{\frac{1}{2}}\left(\Gamma^{-\frac{1}{2}}f(\Theta)\Gamma^{\frac{1}{2}}\right)^*
\end{aligned}
$$

that is $\Gamma^{-\frac{1}{2}}f(\Theta)\Gamma^{\frac{1}{2}}$ is normal.

(iii) As $f(z) = f^*(z)$ in a neighborhood of $[-1, 0]$, if f is real when restricted to $[-1, 0]$, the last part of the assertion now follows directly by the above computation for (b) and by a similar calculation for (a) taking into account (3.8). □

4. Analysis of the linear model

In this section we will investigate ergodic properties of the class of linear models presented in Section 2.

The study of the spectrum of Θ has already emphasized that if $M \notin (1+\frac{K}{1+\kappa}, 1+\frac{K}{\kappa})$, then it is present a point eigenvalue, hence the motion has a periodic component. Clearly, in such a situation the system will exhibit a periodic component (and thus will not be mixing), unless the point eigenvalue corresponds to the ground state. To simplify matters we will restrict our discussion to the case $M \in (1+\frac{K}{1+\kappa}, 1+\frac{K}{\kappa})$ and split up our analysis in two cases: $\mu = -\kappa < 0$ and $\mu = 0$.

4.1. **Asymptotic abelianess.**

Here we analyze the properties of asymptotic abelianess of (suitable CCR subalgebras of) the CCR C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$. Because of the possibility of an infrared divergence, due to the presence of the spectrum of Θ up to 0^-, we distinguish the case with $\mu = 0$ from the case with $\mu < 0$ where no infrared divergence is present.

Proposition 4.1. *The following assertions hold.*

(i) *For each $\mu < 0$ the C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ is asymptotically abelian w.r.t. the dynamics generated by $t \to W(T^\mu_t v)$.*

(ii) *If $\mu = 0$ the C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ is also asymptotically abelian w.r.t. the dynamics generated by $t \to W(T^0_t v)$.*

Proof. Setting

$$S := \begin{pmatrix} \Gamma & 0 \\ 0 & I \end{pmatrix}, \quad \tilde{S} := \begin{pmatrix} I & 0 \\ 0 & \Gamma^{-1} \end{pmatrix}, \tag{4.1}$$

we compute $\sigma(u, T_t^\mu v)$ for general elements u and v in $L^2(\mathbb{Z})^2$ (or in $W^2(\mathbb{Z})^2$ when $\mu = 0$). By Proposition B.2 and (B.1) it follows

$$\sigma(u, T_t^\mu v) = \frac{1}{2\pi} \sum_{m,j=1}^{2} \int_{-1}^{0} A_{mj}(x+\mu, t) r_{(\tilde{S}u)^m,(Sv)^j}(x) dx$$

where $A_{mj}(z+\mu, t)$, $m, j = 1, 2$ are (except a sign) one of the functions $h_p(z+\mu, t)$, $h_d(z+\mu, t)$ or $(z+\mu)h_d(z+\mu, t)$ given in Appendix B. After the change of variable $\xi^2 = -(x+\mu)$, we obtain

$$\sigma(u, T_t^\mu v) = \frac{1}{\pi} \sum_{m,j=1}^{2} \int_{\sqrt{-\mu}}^{\sqrt{1-\mu}} a_{mj}(\xi t) f_{mj}(\xi) r_{\tilde{S}u^m, Sv^j}(-(\xi^2+\mu)) d\xi$$

where $a_{mj}(\xi t)$ is $\sin(\xi t)$ for the diagonal elements and $\cos(\xi t)$ for the off-diagonal ones whereas $f_{mj}(\xi)$ is always ξ except for $(m, j) = (2, 1)$ when it is equal to one. Finally, for every $m, j \in \{1, 2\}$ and $\mu \le 0$, we have the estimate

$$\begin{aligned} &2\int_{\sqrt{-\mu}}^{\sqrt{1-\mu}} d\xi |f_{mj}(\xi) r_{(\tilde{S}u)^m,(Sv)^j}(-(\xi^2+\mu))| \\ &\le \max\{\sqrt{1-\mu}, 1\} \int_{-1}^{0} dx \frac{|r_{(\tilde{S}u)^m,(Sv)^j}(x)|}{\sqrt{-(x+\mu)}}. \end{aligned}$$

(i) From (B.2) we obtain

$$\int_{-1}^{0} dx \frac{|r_{(\tilde{S})u^m,(Sv)^j}(x)|}{\sqrt{-(x+\mu)}} \le \frac{C_1 a}{\sqrt{-\mu}(1-a)} \|u\|_{L^2} \|v\|_{L^2}.$$

So that $\sigma(u, T_t^\mu v)$ is the combination of four pieces each of them being proportional to the real or imaginary part of the Fourier transform of a summable function. We immediately conclude, by Riemann–Lebesgue Lemma,

$$\lim_{t\to\pm\infty} \sigma(u, T_t^\mu v) = 0.$$

Now, as

$$\| [W(u), W(T_t^\mu v)] \| = 2 \sin |\sigma(u, T_t^\mu v)|,$$

we have proved the assertion for a total set in $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$. The proof follows by a 3ε-argument.[5]

[5]Note that this result, as well as analogous results contained in this section, is well known (see, e.g., [19, 29]) and can be obtained in a softer way (that is, the precise knowledge of the spectral measure it is not needed).

(ii) The assertion follows by (B.3). □

4.2. **Quasi-free states: case with $\mu < 0$.**

Here we exhibit a class of quasi-free states which are invariant on the CCR C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ and mixing for the dynamics given by the Bogoliubov automorphisms (2.3).

In order to construct quasi-free states, we start by their two-point functions[6] and we adopt the formalism developed in [1, 2] for a pure matter of convenience (see below). In this way, it is allowed to compute two-point functions for general complex combinations of (unbounded) fields and conjugate momenta. In order to recover the corresponding quasi-free state on the Weyl algebra it is enough to restrict the two-point function to real elements.

We consider the *self-dual CCR algebra* $\mathfrak{A}(L^2(-1,1)^2, \tilde{C}, \gamma)$ where we are using the notation $L^2(-1,1)^2 := L^2(-1,1) \oplus L^2(-1,1)$. The antilinear operator $\tilde{C} := C \oplus C$ is the natural conjugation (see (3.1)) for the phase-space $L^2(-1,1)^2$ and the bilinear form $\gamma = 2i\sigma$ given by

$$\gamma(u,v) := \left\langle u, \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix} v \right\rangle$$

leads to the natural commutation function for (complex combinations of) fields and momenta arising from the sympletic form (3.2).

Proposition 4.2. *Let $F(z)$ be a complex valued function which is analytic on a neighborhood of the spectrum of Θ_μ and such that*

(i) $F(z) > 0$,
(ii) $-zF(z)^2 \geq \frac{1}{4}$

when restricted to the spectrum of $\Theta_\mu \equiv [-1+\mu, \mu]$.

Then the two-point function given by

$$B_F(u,v) = \left\langle u, \begin{pmatrix} F(\Theta_\mu)\Gamma & \frac{i}{2}I \\ -\frac{i}{2}I & -\Gamma^{-1}\Theta_\mu F(\Theta_\mu) \end{pmatrix} v \right\rangle$$

leads to a quasi-free state, invariant for $\alpha_t^{\mu,0}$, on the self-dual CCR-algebra $\mathfrak{A}(L^2(-1,1)^2, \tilde{C}, \gamma)$.

---

[6]If $B(u,v)$ is a two-point function (see e.g. [1, 2, 7, 22, 34]), then the associated quasi-free state ω is uniquely determined, on the CCR algebra, by

$$\omega(W(v)) = e^{-\frac{1}{2}B(v,v)}.$$

Proof. If one considers a two-point function as above, the condition (i) implies $F^*(z) = F(z)$ on a neighborhood of the spectrum of Θ_μ. In this case, by Proposition 3.1, one immediately concludes that $CF(\Theta_\mu)C = F(\Theta_\mu)$ and $\Gamma^{-\frac{1}{2}}F(\Theta_\mu)\Gamma^{\frac{1}{2}}$, $\Gamma^{-\frac{1}{2}}\Theta_\mu F(\Theta_\mu)\Gamma^{\frac{1}{2}}$ are both self-adjoint. Now, as the conjugation commutes with any analytical functional calculus of Θ_μ as above, we have easily

$$B_F(u,v) - B_F(\tilde{C}v, \tilde{C}u) = \gamma(u,v),$$

thus the two point function is compatible with the commutation relations.

It remains to verify the positivity of ω which reduces to $B_F(u,u) \geq 0$ for any $u \in L^2(-1,1)^2$, see [1, 2]. The last condition is equivalent to

$$\left\langle u, \begin{pmatrix} \Gamma^{-\frac{1}{2}}F(\Theta_\mu)\Gamma^{\frac{1}{2}} & \frac{i}{2}I \\ -\frac{i}{2}I & -\Gamma^{-\frac{1}{2}}\Theta_\mu F(\Theta_\mu)\Gamma^{\frac{1}{2}} \end{pmatrix} u \right\rangle \geq 0$$

or, alternatively, by using (i),

$$\begin{aligned} &\|[\Gamma^{-\frac{1}{2}}F(\Theta_\mu)\Gamma^{\frac{1}{2}}]^{\frac{1}{2}}u_1 + \frac{i}{2}[\Gamma^{-\frac{1}{2}}F(\Theta_\mu)\Gamma^{\frac{1}{2}}]^{-\frac{1}{2}}u_2\|^2 \\ -&\frac{1}{4}\langle u_2, \Gamma^{-\frac{1}{2}}F(\Theta_\mu)^{-1}\Gamma^{\frac{1}{2}}u_2\rangle - \langle u_2, \Gamma^{-\frac{1}{2}}\Theta_\mu F(\Theta_\mu)\Gamma^{\frac{1}{2}}u_2\rangle \geq 0, \end{aligned}$$

which is implied by $\frac{1}{4}\Gamma^{-\frac{1}{2}}F(\Theta_\mu)^{-1}\Gamma^{\frac{1}{2}} \leq -\Gamma^{-\frac{1}{2}}\Theta_\mu F(\Theta_\mu)\Gamma^{\frac{1}{2}}$. This last condition, taking (i) into account, is equivalent to (ii).

The invariance condition is satisfied since it is equivalent to

$$\begin{aligned} 0 = \Lambda_\mu &\begin{pmatrix} F(\Theta_\mu)\Gamma & \frac{i}{2}I \\ -\frac{i}{2}I & -\Gamma_{-1}\Theta_\mu F(\Theta_\mu) \end{pmatrix} \\ &+ \begin{pmatrix} F(\Theta_\mu)\Gamma & \frac{i}{2}I \\ -\frac{i}{2}I & -\Gamma_{-1}\Theta_\mu F(\Theta_\mu) \end{pmatrix} \Lambda_\mu^* . \end{aligned}$$

□

A straightforward computation shows that, if we have a quasi-free state on $\mathfrak{A}(L^2(-1,1)^2, \tilde{C}, \gamma)$, the corresponding quasi-free state on the CCR C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ can be obtained by restricting

$$\begin{aligned} \langle u, B_F v\rangle :=& B_F(u,v) - \frac{1}{2}\gamma(u,v) \\ \equiv& \left\langle u, \begin{pmatrix} F(\Theta_\mu)\Gamma & 0 \\ 0 & -\Gamma^{-1}\Theta_\mu F(\Theta_\mu) \end{pmatrix} v \right\rangle \end{aligned} \tag{4.2}$$

to the "real" elements f (that is $\tilde{C}f = f$) of $L^2(-1,1)^2$ (here we have identified elements in $L^2_{\mathbb{R}}(\mathbb{Z})$ with their Fourier transform). In fact, it is straightforward to verify that all conditions given in [34], pag 173, are automatically satisfied.

We now analyze the mixing properties of the quasi-free states on the CCR-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ given by the above two-point function.

Proposition 4.3. *Let ω_F be the quasi-free state on the CCR C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ given by the two-point function (4.2) relative to the analytic function F.*

Then $(\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)},\alpha^{\mu,0},\omega_F)$ is a mixing QDS.

Proof. The proof proceeds as the part (i) of Proposition 4.1.

By density it is enough to compute

$$\lim_{t\to\pm\infty}\omega_F(W(u)W(T^\mu_t v))$$

for $u,v\in L^2_{\mathbb{R}}(\mathbb{Z})^2$. We get

$$\begin{aligned}&\omega_F(W(u)W(T^\mu_t v))=e^{-i\sigma(u,T^\mu_t v)}\omega_F(W(u+T^\mu_t v))\\ =&\omega_F(W(u))\omega_F(W(v))e^{-(i\sigma(u,T^\mu_t v)+\frac{1}{2}\langle u,B_F T^\mu_t v\rangle)+\frac{1}{2}\langle v,B_F T^\mu_{-t}u\rangle)}.\end{aligned}$$

Taking into account Proposition B.2 we obtain, after an elementary change of variable,

$$\begin{aligned}&\langle u,B_F T^\mu_t v\rangle\\ &=\sum_{m,j=1}^{2}\int_{\sqrt{-\mu}}^{\sqrt{1-\mu}}F(-\xi^2)a_{mj}(\xi t)f_{mj}(\xi)r_{(\tilde{S}u)^m,(Sv)^j}(-(\xi^2+\mu))\xi d\xi\end{aligned}$$

where $a_{mj}(\xi t)$ are the cosine function for the diagonal elements and the sine function for the off-diagonal elements, whereas $f_{mj}(\xi)$ are all bounded (the linear maps S, $\tilde{S}$ are as in (4.1)). The proof easily follows from (B.2) by Riemann-Lebesgue Lemma. □

Remark 4.4. *For the sake of completeness, we remark that, on the CCR C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$, quasi-free states given by unbounded functional calculi of the operator Θ_μ can also be defined. These quasi-free states leads to QDSs which are mixing as well.*

For this generalization see Proposition 4.5 and Proposition 4.6.

4.3. **Quasi-free states: case with $\mu=0$.**

Now we describe quasi-free states on the CCR algebra in presence of infrared divergence, that is when $\mu=0$. In this case, condition (ii) of Proposition 4.2 contradicts the hypothesis that F be analytic in a neighborhood of $[-1,0]$. To overcome this difficulty, we need weaker conditions than those contained in Proposition 4.2. This can be done by paying the price of restricting the CCR C^*-algebra to be $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$.

Proposition 4.5. *Let $F(z)$ be a complex valued function, analytic on a neighborhood of $[-1,0)$, which satisfies*

(i) $F(z)>0$ *on* $[-1,0)$,
(ii) $-zF(z)^2\geq\frac{1}{4}$ *on* $[-1,0)$,
(iii) $\int_{-1}^{0}dx\sqrt{-x}F(x)<+\infty$.

Then the two-point function given by

$$B_F(u,v):=\frac{1}{\pi}\int_{-1}^{0}dxF(x)[r_{u^1,\Gamma v^1}(x)-xr_{\Gamma^{-1}u^2,v^2}(x)]$$

gives rise to a quasi-free state ω_F on the CCR C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ which is invariant for the dynamics $\alpha^{0,0}_t$.

Furthermore, $(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)},\alpha^{0,0},\omega_F)$ is a QDS.

Proof. We report only a sketch of the proof and leave the details to the reader.

Thanks to the conditions (i)–(iii), the above two-point function is well defined (i.e. finite) and satisfies, on $(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)$ all conditions of pag 173 of [34]. So it defines a regular quasi-free state on $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$.

In order to prove the invariance property, it is enough to prove it for the Weyl operators. But this easily follows taking into account that B_F arises also from a (unbounded) functional calculus of Θ, see Proposition B.2.[7]

Concerning the last assertion, by invariance it is enough to show that, for $u,v\in W^2_{\mathbb{R}}(\mathbb{Z})^2$, the function $t\to B_F(u,T_tv)$ is continuous. But this is a simple consequence of condition (iii) by the application of Lebesgue dominated convergence Theorem. □

We now analyze the mixing properties of quasi-free states described, in presence of infrared divergence, by Proposition 4.5.

Proposition 4.6. *Suppose that F satisfies the properties of Proposition 4.5.*

Then the QDS $(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)},\alpha^{0,0},\omega_F)$ is mixing.

---

[7] One could exhibit a more direct proof of this fact as, by condition (iii),

$$B_F(T_tu,T_tv)=\lim_{\mu\to0^-}\left\langle T^\mu_tu,\begin{pmatrix}F(\Theta_\mu)\Gamma & 0\\ 0 & -\Gamma^1\Theta_\mu F(\Theta_\mu)\end{pmatrix}T^\mu_tv\right\rangle.$$

The assertion now follows by Lebesgue dominated convergence Theorem taking into account the invariance property of the r.h.s..

Proof. As

$$\begin{aligned}&\omega_F(W(u)W(T_t^0v))=e^{-i\sigma(u,T_t^0v)}\omega_F(W(u+T_t^0v))\\ =&\omega_F(W(u))\omega_F(W(v))e^{-i\sigma(u,T_t^0v)+\frac{1}{2}B_F(u,T_t^0v)+\frac{1}{2}B_F(v,T_{-t}^0u)}\end{aligned}$$

for $u,v\in W^2_{\mathbb{R}}(\mathbb{Z})^2$, it is enough to show that

$$\lim_{t\to\pm\infty}B_F(u,T_t^0v)=0\,.$$

By part (ii) of Proposition B.2 we obtain

$$B_F(u,T_t^0v)=\sum_{m,j=1}^{2}\int_0^1 F(-\xi^2)a_{mj}(\xi t)f_{mj}(\xi)r_{(\tilde{S}u)^m,(Sv)^j}(-\xi^2)\xi d\xi$$

where $a_{mj}(\xi t)$ are as in Proposition 4.3 and $f_{mj}(\xi)$ are all continuous bounded functions. Now, in order to apply Riemann-Lebesgue Lemma we should estimate

$$\begin{aligned}&\int_0^1 F(-\xi^2)|\xi f_{mj}(\xi)r_{(\tilde{S}u)^m,(Sv)^j}(-\xi^2)|d\xi\\ &\leq c\int_{-1}^0\sqrt{-x}F(x)\frac{|r_{(\tilde{S}u)^m,(Sv)^j}(x)|}{\sqrt{-x}}dx\end{aligned}$$

where by (B.3), the r.h.s. is bounded, keeping in mind condition (iii) of Proposition 4.5. □

4.4. **The KMS states.**

Here we check the temperature states for our model making a distinction between the case $\mu<0$ and the case $\mu=0$ where infrared divergences are present. We adopt, for the KMS boundary condition, the definition contained in [32], pag. 133.

We fix our attention on the complex function

$$F_{\mu,\beta}(z):=\frac{\coth(\frac{\beta}{2}\sqrt{-(z+\mu)})}{\sqrt{-(z+\mu)}}$$

where the square root is defined according to Appendix A.

Proposition 4.7. *Let $\mu\leq 0$, $\beta>0$ be fixed and consider the two-point function*

$$B_{\mu,\beta}(u,v):=\frac{1}{\pi}\int_{-1}^0 dxF_{\mu,\beta}(x)[r_{u^1,\Gamma v^1}(x)-(x+\mu)r_{\Gamma^{-1}u^2,v^2}(x)]\,. \tag{4.3}$$

(i) *If* $\mu<0$ *and* $u,v\in L^2(\mathbb{Z})^2$, *then the two-point function (4.3) gives rise to a KMS state* $\omega_{\mu,\beta}$ *at the inverse temperature* β *for the CCR* C^**-algebra* $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ *w.r.t. the dynamics* $\alpha^{\mu,0}_t$.

(ii) *If* $\mu=0$ *and* $u,v\in W^2(\mathbb{Z})^2$, *then the two-point function (4.3) gives rise again to a KMS state* $\omega_{0,\beta}$ *at the inverse temperature* β *for the CCR* C^**-algebra* $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ *w.r.t. the dynamics* $\alpha^{0,0}_t$.

Proof. It is well known that the analytical functional calculus relative to the function

$$F(z)=\frac{\coth(\frac{\beta}{2}\sqrt{-z})}{\sqrt{-z}}$$

gives rise a KMS state for the dynamics (2.4). This can be seen, e.g., by computing the KMS states for finite systems (see [34], Lemma 5) and taking the limit (see also [19], Section 6).

(i) As $R_{A+\lambda I}(z)=R_A(z-\lambda)$, a simple change of variable gives the assertion.

(ii) The assertion follows as

$$\int_{-1}^{0}dx\sqrt{-x}F_{0,\beta}(x)<+\infty\,.$$

□

It is straightforward to verify that the complex function

$$F(z)=\frac{1}{\sqrt{-z}}$$

leads to ground states for our systems.

We leave to the reader the proof of the following

Proposition 4.8. *Let* $\mu\leq 0$ *be fixed and consider the two-point function*

$$B_{\mu,\infty}(u,v):=\frac{1}{\pi}\int_{-1}^{0}dx\left[\frac{r_{u^1,\Gamma v^1}(x)}{\sqrt{-(x+\mu)}}+\sqrt{-(x+\mu)}\,r_{\Gamma^{-1}u^2,v^2}(x)\right].\tag{4.4}$$

(i) *If* $\mu<0$ *and* $u,v\in L^2(\mathbb{Z})^2$, *then the two-point function (4.4) gives rise to a ground state* $\omega_{\mu,\infty}$ *for the CCR* C^**-algebra* $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ *w.r.t. the dynamics* $\alpha^{\mu,0}_t$.

(ii) *If* $\mu=0$ *and* $u,v\in W^2(\mathbb{Z})^2$, *then the two-point function (4.4) gives rise again to a ground state* $\omega_{0,\infty}$ *for the CCR* C^**-algebra* $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ *w.r.t. the dynamics* $\alpha^{0,0}_t$.

As an immediate corollary we have the following

Proposition 4.9. *Let $\mu < 0$, $\beta \in (0, +\infty) \cup \{\infty\}$. Then both QDSs $(\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}, \alpha_t^{\mu,0}, \omega_{\mu,\beta})$ and $(\overline{\mathfrak{A}(W^{2,2}_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}, \alpha_t^{0,0}, \omega_{0,\beta})$ are mixing.*

Proof. The proof is an immediate consequence of Proposition 4.3 and Proposition 4.6 respectively. ∎

5. The non linear case

In this chapter we want to extend the results contained in the first part of the paper to some non harmonic cases.

Namely, we would like to consider the case when some particles are subject to a non quadratic external potential. Since the main purpose here is to exhibit non linear models for which the ergodic properties can be completely investigated, we will restrict ourselves to the case with a bounded non quadratic external potential acting only on a particle, for instance the 0^{th}-particle.

5.1. **The non linear perturbation.**

In order to treat the time evolution in presence of a non harmonic external potential acting on the 0^{th}-particle, we need to enlarge the CCR C^*-algebra $\mathfrak{A}$ under consideration since, in general, it will be no longer invariant for the new dynamics. This is done in a standard way by considering W^*-dynamical systems. Such dynamical systems arise from the GNS construction relative to any regular state (say ω) such as the quasi-free states considered in the first part of this paper. Thus, starting by the QDS $(\mathfrak{A}, \alpha_t^0, \omega)$ with ω regular, we obtain naturally a W^* dynamical system $(\pi_\omega(\mathfrak{A})'', \alpha_t^0)$ where the linear dynamics α_t^0 is generated (in Heisenberg picture) by a strongly continuous one parameter unitary group:[8]

$$\pi_\omega(\alpha_t^0 A) = U_t \pi_\omega(A) U_t^{-1}.$$

Since we consider a bounded perturbation of the dynamics according to [7], Proposition 5.4.1, for a fixed element $P \in \pi_\omega(\mathfrak{A})''$ the perturbed dynamics will be the unique solution of

$$\alpha_t^P A = \alpha_t^0 A + i \int_0^t ds \alpha_t^P [P,\, \alpha_{t-s}^0 A] \tag{5.1}$$

[8]To simplify notations, we will suppress the index μ in the dynamics since this will not create ambiguities in the sequel. Thus we write α_t^0 instead of $\alpha_t^{\mu,0}$.

where α_t^0 denotes the linear dynamics extended to any element $A \in \pi_\omega(\mathfrak{A})''$. In our case P is precisely the non quadratic part of the external potential acting on the 0^{th}-particle. It will be expressed in the form

$$P \equiv V(q_0) := \int_{\mathbb{R}} \pi_\omega(W(\lambda e_{0,1}))\nu(d\lambda)\,, \tag{5.2}$$

where ν is any finite complex Radon measure on $\mathbb{R}$ with real Fourier transform. The last property translates as

$$\nu \circ r = \overline{\nu} \tag{5.3}$$

where r is the mirror reflection on the real line and $\overline{\nu}$ denotes the measure obtained by complex conjugation. The above integral is understood as an integral in the strong operator topology (or, equivalently, in the weak operator topology, see [27], Theorem IV.22) as the functions $\lambda \to \pi_\omega(W(\lambda v))$ are all bounded operator-valued functions defined on a separable Hilbert space (see Section 4 together with [3]) which are continuous in the strong operator topology. Clearly, formula (5.2) is the natural operator corresponding to the classical potential

$$V(x) = \int_{\mathbb{R}} e^{i\lambda x}\nu(d\lambda)\,.$$

In order to study the long time behavior of the system, we start with some preliminary computations relative to (the representative of) the Weyl operators $\pi_\omega(W(v))$. In the sequel, we drop the symbol π_ω if no ambiguity arises.

The following calculation is the starting point of our analysis.

$$\begin{aligned}
[P,\, \alpha_t^0 W(v)] &= \int_{\mathbb{R}} \nu(d\lambda)[W(\lambda e_{0,1}),\, W(T_t v)] \\
&= -2i\int_{\mathbb{R}} \nu(d\lambda) W(\lambda e_{0,1} + T_t v)\sin(\lambda\sigma(e_{0,1}, T_t v)) \\
&= -2i\int_{\mathbb{R}} \nu(d\lambda) W(\lambda e_{0,1} + T_t v)\sin(\lambda\gamma_v(t))
\end{aligned}$$

where

$$\gamma_v(t) := \sigma(e_{0,1}, T_t v) = \frac{1}{2}\langle e_{0,2}, T_t v\rangle\,. \tag{5.4}$$

5.2. **The perturbed dynamics for the Weyl operators.**

We begin by studying the dynamics restricting ourselves to (the representative of) the Weyl operators $W(v)$.

Taking into account the above computations and (5.1), the non linear dynamics on the Weyl operators satisfies the equation

$$\begin{aligned}\alpha_t^P W(v) = & \, W(T_t v) \\ & + \int_0^t ds \int_{\mathbb{R}} \nu(d\lambda) \alpha_s^P W(\lambda e_{0,1} + T_{t-s} v) g(v, t-s, \lambda)\end{aligned} \tag{5.5}$$

where

$$g(v,s,\lambda) := 2 \sin\left(\lambda \gamma_v(s)\right).$$

Setting $\underline{t}^n := (t_n, \ldots, t_1)$ and $\underline{\lambda}^n := (\lambda_n, \ldots, \lambda_1)$, let us consider the series given by

$$\begin{aligned} X_t(v) = W(T_t v) + & \sum_{n=1}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \\ & \times \int_{\mathbb{R}^n} \nu(d\lambda_1) \ldots \nu(d\lambda_n) W(w_n(v, t, \underline{t}^n, \underline{\lambda}^n)) \\ & \times G_n(v, t, \underline{t}^n, \underline{\lambda}^n) \end{aligned} \tag{5.6}$$

where,

$$\begin{aligned} w_0(v,t) :=& T_t v \\ w_{n+1}(v,t,\underline{t}^{n+1},\underline{\lambda}^{n+1}) :=& w_n(\lambda_{n+1} e_{0,1} + T_{t-t_{n+1}} v, t_{n+1}, \underline{t}^n, \underline{\lambda}^n), \\ G_0 :=& 1 \\ G_{n+1}(v,t,\underline{t}^{n+1},\underline{\lambda}^{n+1}) :=& G_n(\lambda_{n+1} e_{0,1} + T_{t-t_{n+1}} v, t_{n+1} \underline{t}^n, \underline{\lambda}^n) \\ & \times g(v, t - t_{n+1}, \lambda_{n+1}). \end{aligned}$$

Proposition 5.1. *The series given in (5.6) is norm-absolutely summable on bounded sets of* $\mathbb{R}$ *and describes the non linear time evolution for all Weyl operators.*[9]

Namely, if $v \in (W_{\mathbb{R}}^2(\mathbb{Z})^2, \sigma)$, *then*

$$X_t(v) = \alpha_t^P \pi_\omega(W(v)).$$

Proof. We start by recalling that, if we have a bounded function

$$\lambda \in X \to A(\lambda) \in \mathcal{B}(\mathcal{H})$$

[9]If $\mu < 0$, that is when the states under considerations are well defined on all of $\mathfrak{A}(L_{\mathbb{R}}^2(\mathbb{Z})^2, \sigma)$, the series (5.6) defines the non linear time evolution for each $v \in L_{\mathbb{R}}^2(\mathbb{Z})^2$. This could lead to a slight generalization of some of results in the sequel. To simplify matters we will not indulge on such a subtlety.

between a σ-finite measure space (X,μ) and the set of all the bounded linear operators on a separable Hilbert space $\mathcal{H}$, which is measurable w.r.t. the Borel field generated by the strong operator topology, it is a standard result that

$$\left\|\int_X A(\lambda)\mu(d\lambda)\right\| \leq \int_X \|A(\lambda)\||\mu|(d\lambda).$$

Since, for states ω here considered, $\mathcal{H}_\omega$ is separable (see [3]), we have $\|W(v)\|=1$, $\|G_n\|_\infty \leq 2^n$ and

$$\begin{aligned}\|X_t(v)\| \leq & 1+\sum_{n=1}^{\infty}(2|\nu|(\mathbb{R}))^n\int_0^{|t|}dt_n\cdots\int_0^{|t_2|}dt_1\\ =&\sum_{n=0}^{\infty}\frac{(2|t|\|\nu\|)^n}{n!}=e^{2|t|\|\nu\|}\,. \end{aligned} \tag{5.7}$$

As the series is norm-absolutely summable, the proof now follows by substituting (5.6) in (5.5) having taken into account the unicity of the solution of (5.5) (see [7]). □

By induction it is easy to verify that

$$\begin{aligned} &w_n(v,t,\underline{t}^n,\underline{\lambda}^n)=\sum_{k=1}^{n}\lambda_k T_{t_{k+1}}e_{0,1}+T_tv,\\ &G_n(v,t,\underline{t}^n,\underline{\lambda}^n)=g(v,t-t_n,\lambda_n)\\ &\qquad\times\prod_{k=1}^{n-1}g\left(T_{-t_{k+1}}\left[\sum_{j=k+1}^{n}T_{t_j}e_{0,1}+T_tv\right],t_{k+1}-t_k,\lambda_k\right). \end{aligned} \tag{5.8}$$

As a final remark let us notice that, using formulae (5.8) and the change of variables $s_k=t-t_k$, the dynamics can be expressed in the more convenient form

$$\alpha_t(W(v))=W(T_tv)+\sum_{n=1}^{\infty}\int_0^t dt_n\int_{t_n}^t dt_{n-1}\cdots\int_{t_2}^t dt_1 \tag{5.9}$$

$$\times\int_{\mathbb{R}^n}\nu(d\lambda_1)\ldots\nu(d\lambda_n)W\left(T_t\hat{w}_n(v,\underline{t}^n,\underline{\lambda}^n)\right)\hat{G}_n(v,\underline{t}^n,\underline{\lambda}^n) \tag{5.10}$$

where, in the above formula,

$$\begin{aligned}\hat{w}_n(v,\underline{t}^n,\underline{\lambda}^n)=&T_{-t}w_n(v,t,t-t_1,\ldots,t-t_n,\underline{\lambda}^n)\\ \equiv&\sum_{k=1}^{n}\lambda_kT_{-t_k}e_{0,1}+v\end{aligned} \tag{5.11}$$

$$\begin{aligned}\hat{G}_n(v,\underline{t}^n,\underline{\lambda}^n) =&G_n(v,t,t-t_1,\dots,t-t_n,\underline{\lambda}^n)\\ \equiv&2^n\prod_{k=1}^n \sin\left(\frac{\lambda_k}{2}\langle e_{0,2}, \sum_{j=k+1}^n \lambda_j T_{(t_k-t_j)}e_{0,1}+T_{t_k}v\rangle\right).\end{aligned} \tag{5.12}$$

We conclude by noticing that $\hat{w}_n$, as well as $\hat{G}_n$, do not depend on t.

6. Convergence to the equilibrium

In this section we investigate the long time behavior of dynamical systems arising from the considerations of the previous section. We begin by specifying the set of states we are interested in.

We consider states ω on $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ such that the GNS representation π_ω acts on a separable Hilbert space $\mathcal{H}_\omega$. We suppose also that the function

$$v\in W^2_{\mathbb{R}}(\mathbb{Z})^2\to\pi_\omega(W(v))\in\mathcal{U}(\mathcal{H}_\omega)$$

is a Borel one, when the unitary group $\mathcal{U}(\mathcal{H}_\omega)$ is equipped with the weak (or equivalently the strong) operator topology.

We note that all quasi-free states considered in the sequel satisfy the above conditions, see Proposition B.2, having taken into account that the Hilbert space $\mathcal{H}_\omega$ of the GNS construction relative to such ω is separable, [3]. In this case, as both functions $\lambda\to\pi_\omega(W(\lambda v))$ and $t\to\pi_\omega(W(T_t v))$ are Borel, hence continuous,[10] the state ω is regular and the dynamical system $(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)},\alpha^0_t,\omega)$ is indeed a QDS according to the definition contained in the introduction.

We assume that ω is invariant w.r.t. α^0_t, even though part of our analysis applies also to non invariant states.

Finally, for reasons which will appear clear below, we deal only with the CCR algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$. In fact, if $\mu<0$ (that is without infrared divergences), our analysis applies to the (larger) CCR algebra $\overline{\mathfrak{A}(W^1_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$, see Proposition 4.1 and Proposition 4.3 together with Lemma B.3.

Under the above conditions, the analysis developed in Section 5 applies.

[10] It is straightforward to show that, under the above conditions, the function $v\to\pi_\omega(W(v))$ is indeed continuous, when the unitary group $\mathcal{U}(\mathcal{H}_\omega)$ is equipped with the weak operator topology (see [23], Proposition 5), as the canonical commutation relation can be easily enlarged to a continuous group operation on the Polish space $W^2_{\mathbb{R}}(\mathbb{Z})^2\times\mathbb{T}$.

We consider, on the representation relative to the fixed state ω, the dynamics α_t^P arising from the non linear perturbation (5.2) where the measure ν satisfies

$$\|\nu\| := \sup_{n\in\mathbb{N}} \frac{1}{(n-1)!} \int_{\mathbb{R}} |\nu|(d\lambda)|\lambda|^n < \frac{1}{2\|\gamma_{e_{0,1}}\|}, \tag{6.1}$$

if it is not otherwise specified.

For a bounded measure ν on the real line, the seminorm defined in (6.1) is finite if and only if the potential V can be extended to an analytic complex function on the open strip of size 1 around the real axis. One could consider more general situations by introducing the norms

$$\|\nu\|_\theta := \int_{\mathbb{R}} e^{|\lambda|\theta} |\nu|(d\lambda).$$

These norms allow us to treat potentials analytic in an arbitrarily small strip around the real axis, provided they are small enough; we choose not to indulge on this generalization in order to simplify the exposition. At the moment it is unclear what can be said in general for non analytic potentials.

6.1. **The C^*-algebra generated by measures.**

In order to study the asymptotic properties of the systems at hand we note that the W^*-algebra $\pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''$ turns out to be too large. However we can deal with a suitable C^*-subalgebra $\mathfrak{M}_\omega \subset \pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''$ which is large enough to be invariant with respect to the non linear dynamics and to contain all the relevant observables (i.e. all Weyl operators), but sufficiently small to allow uniform estimates in time.

We start with triples $\mathfrak{m} := (m, h, \mu)$ where $m \in \mathbb{N}$, $h : \mathbb{R}^m \to W^2_{\mathbb{R}}(\mathbb{Z})^2$ is a Borel function and μ is a "suitable" complex measure on $\mathbb{R}^m$.

More precisely, given the set

$$M := \{(m, h, \mu) \mid \exists\varepsilon > 0 : \int_{\mathbb{R}^m} e^{\varepsilon\|h(x)\|_{W^2(\mathbb{Z})^2}} |\mu|(dx) < \infty\},$$

we can define, for each $\mathfrak{m} = (m, h, \mu) \in \mathcal{M}_\omega$,

$$W(\mathfrak{m}) = \int_{\mathbb{R}^m} \pi_\omega(W(h(x))\mu(dx),$$

where the integrals are meant w.r.t the strong (equivalently weak) operator topology on $\pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''$. Next, we consider the following linear set of operators[11]

$$\mathcal{M}_\omega := \operatorname{span}\left\{\int_{\mathbb{R}^m} \pi_\omega(W(h(x)))\mu(dx)\ ;(m,h,\mu)\in M\right\}\ .$$

Lemma 6.1. *$\mathcal{M}_\omega$ is a *-subalgebra of $\pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''$.*

Proof. The closure w.r.t. the sum is trivial. The adjoint follows since it is easy to check that

$$\left[\int_{\mathbb{R}^n} W(h(x))\mu(dx)\right]^* = \int_{\mathbb{R}^n} W(-h(x))\bar{\mu}(dx)$$

where $\bar{\mu}$ is the complex conjugate of the measure μ. The closure w.r.t. the product easily follows from Fubini Theorem. Indeed

$$\begin{aligned}&\left[\int_{\mathbb{R}^m} W(h_1(x))d\mu_1(x)\right]\left[\int_{\mathbb{R}^n} W(h_2(x))d\mu_2(x)\right]\\ &=\int_{\mathbb{R}^{n+m}} W(h_1(x)+h_2(y))d\mu_3(x,y)\end{aligned}$$

where $d\mu_3(x,y)=e^{-i\sigma(h_1(x),\,h_2(y))}d\mu_1(x)d\mu_2(y)$. □

Let now $\mathfrak{M}_\omega$ be the C^*-algebra generated by $\mathcal{M}_\omega$.

Proposition 6.2. *$\mathfrak{M}_\omega$ is globally stable for the one-parameter group of automorphisms α^P_t of $\mathcal{B}(\mathcal{H}_\omega)$.*

Proof. Let $\int W(h(x))d\mu(x)\in\mathcal{M}_\omega$, then

$$\begin{aligned}\alpha^P_t\int W(h(x))d\mu(x) &= \int \alpha^P_t W(h(x))d\mu(x)\\ &=\sum_{n=0}^{\infty}\int W(w_n(h(x),t,\underline{t}^n,\underline{\lambda}^n))G_n(h(x),t,\underline{t}^n,\underline{\lambda}^n)\end{aligned}$$

since, by estimate (5.7), the series converges in norm. Now, the n-th element of the series is defined by $\tilde{h}(x,\underline{t}^n,\underline{\lambda}^n):=w_n(h(x),t,\underline{t}^n,\underline{\lambda}^n)$ together with the measure (except a sign when $t<0$)

$$d\tilde{\mu} := \chi_{\Delta^n_t}(\underline{t}^n)G_n(h(x),t,\underline{t}^n,\underline{\lambda}^n)d\mu(x)d\nu(\lambda_1)\dots d\nu(\lambda_n)dt_1\dots dt_n,$$

where $\Delta^n_t=\{(t_1,\dots,t_n)\in\mathbb{R}^n \mid t\geq t_1\geq\dots\geq t_n\geq o\}$.

Since T_t is continuous as an operator from $W^2(\mathbb{Z})^2$ to itself, it follows by (5.11) that there exists c_t such that $\|\tilde{h}\|_{W^2}\leq c_t\{\|h\|_{W^2}+$

[11] Here we are dealing with a *-algebra of measures which is somewhat similar to that usually considered, see e.g. [17]. However, it may be possible to consider a slightly larger C^*-algebra by allowing directly measures on $W^2(\mathbb{Z})^2$. Nevertheless, this would entail several technical problems without adding much to the present results. Thus we have chosen not to pursue such a generalization.

$\sum_{i=1}^{n}|\lambda_i|\}$. Thus, since $\int e^{\varepsilon\|h\|}d\mu<\infty$ for some $\varepsilon>0$, then, setting $\varepsilon_1=\min\{\varepsilon c_t^{-1},\frac{1}{2}\}$, $\int e^{\varepsilon_1\|\tilde{h}\|_{W^2}}d\tilde{\mu}<\infty$, that is $(2n+m,\tilde{h},\tilde{\mu})\in M$. Namely, the above series is made of elements of $\mathcal{M}_\omega$ and converges in norm so it defines an element of $\mathfrak{M}_\omega$.

The statement follows by usual density arguments. □

Next, we show that the Møller morphisms [7] exist also in our situations. Even though this fact is not needed in the sequel, we report it for the sake of completeness.

We start by defining on $\mathfrak{M}_\omega$

$$\gamma_t^{\omega,P}(A):=\alpha_{-t}^{P}\alpha_t^{0}A.$$

Only in this case we assume that the measure ν satisfies the weaker condition

$$\int_{\mathbb{R}}|\lambda||\nu|(d\lambda)<\infty$$

instead of condition (6.1).

Theorem 6.3. *Both limits*

$$\lim_{t\to\pm\infty}\gamma_t^{\omega,P}(A)=:\gamma_\pm^{\omega,P}(A)$$

exist pointwise in the norm topology of $\mathfrak{M}_\omega$.

Proof. Suppose that $|s|<|t|$. By (5.1) we obtain

$$\|\gamma_t^{\omega,P}(A)-\gamma_s^{\omega,P}(A)\|\leq\int_{|s|}^{|t|}d\tau\|[P,\alpha_t^{0}A]\|\,. \tag{6.2}$$

We note that it is enough to prove the statement for the generators $\{W(\mathfrak{m})\}_{\mathfrak{m}\in M}$ of $\mathfrak{M}_\omega$ and, in this case, formula (6.2) reads

$$\begin{aligned}&\|\gamma_t^{\omega,P}(W(\mathfrak{m}))-\gamma_s^{\omega,P}(W(\mathfrak{m}))\|\\ \leq&2\int_{|s|}^{|t|}d\tau\left[\int_{\mathbb{R}^{m+1}}|\nu|(d\lambda)|\mu|(dx)\sin|\lambda\gamma_{h(x)}(\tau)|\right]\,.\end{aligned} \tag{6.3}$$

As, by Lemma B.3,

$$
\begin{aligned}
&\int_{\mathbb{R}^{m+2}} \sin|\lambda\gamma_{h(x)}(t)||\nu|(d\lambda)|\mu|(dx)dt \\
\leq &\int_{\mathbb{R}^{m+2}} |\lambda||\gamma_{h(x)}(t)||\nu|(d\lambda)|\mu|(dx)dt \\
\leq &c_1\int_{\mathbb{R}} |\lambda||\nu|(d\lambda)\int_{\mathbb{R}^m}\|h(x)\|_{W^2(\mathbb{Z})^2}|\mu|(dx) \\
\leq &\frac{c_1}{e\varepsilon}\int_{\mathbb{R}} |\lambda||\nu|(d\lambda)\int_{\mathbb{R}^m}e^{\varepsilon\|h(x)\|_{W^2(\mathbb{Z})^2}}|\mu|(dx)
\end{aligned}
$$

(where ε depends on μ according to the definition of $\mathfrak{m}$), we conclude, by Fubini Theorem, that the function

$$f(t) := \int_{\mathbb{R}^{m+1}} \sin|\lambda\gamma_{h(x)}(t)||\nu|(d\lambda)|\mu|(dx)$$

is summable. The assertion now follows since $f(t)$ is precisely the integrand of the r.h.s. of (6.3). $\square$

6.2. **Asymptotic abelianess.**

Here we check the properties of asymptotic abelianess for the dynamical system $(\mathfrak{M}_\omega, \alpha^P_t)$.

Theorem 6.4. *The dynamical system $(\mathfrak{M}_\omega, \alpha^P_t)$ is asymptotically abelian.*

Proof. By density and the Lebesgue dominated convergence Theorem, it is enough to check that

$$\lim_{t\to\pm\infty}\big[W(u),\alpha^P_t W(v)\big]=0$$

for any choice of element $u, v \in W^2_{\mathbb{R}}(\mathbb{Z})^2$. Using formula (5.10) for the time evolution of the Weyl operators and Lemma C.1, by the Lebesgue dominate convergence Theorem we can exchange the limit with the summation, the proof follows provided that

$$\lim_{t\to\pm\infty}[W(u),W(T_t\hat{w}_n(v,\underline{t}^n,\underline{\lambda}^n))]=0$$

for each $n \in \mathbb{N}$ pointwise w.r.t. $\underline{t}^n$ and $\underline{\lambda}^n$. But this follows as

$$\lim_{t\to\pm\infty}\sigma(u,T_t(\sum_{k=1}^n \lambda_k T_{-t_k}e_{0,1}+v))=0\,.$$

$\square$

In order to study the asymptotic behavior of dynamical systems, we remark the following simple fact.

Consider a state φ in $\pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''_*$. Such a state is given (possibly in a non unique manner) by a density matrix on the Hilbert space $\mathcal{H}_\omega$ of the GNS triple relative to ω.

Remark 6.5. *If the dynamical system* $(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)},\alpha^0_t)$ *is asymptotically abelian and ω is mixing, then*

$$\lim_{t\to\pm\infty}\varphi(\alpha^0_t A)=\omega(A) \tag{6.4}$$

for each $A\in\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$.

6.3. **Long time behavior.**

Let us start by fixing some notations.

Recalling the definition (5.11) and (5.12) for $\hat{w}$ and $\hat{G}$ respectively, we define

$$\begin{aligned}a^{\pm}_n(v):=&\int_0^{\pm\infty}dt_n\int_{t_n}^{\pm\infty}dt_{n-1}\cdots\int_{t_2}^{\pm\infty}dt_1\int_{\mathbb{R}^n}\nu(d\lambda_n)\dots\nu(d\lambda_1)\\ &\times\omega\left(W\left(\hat{w}_n(v,\underline{t}^n,\underline{\lambda}^n)\right)\right)\hat{G}_n(v,\underline{t}^n,\underline{\lambda}^n)\end{aligned} \tag{6.5}$$

for each $v\in W^2_{\mathbb{R}}(\mathbb{Z})^2$, $n\in\mathbb{N}$ and $a^{\pm}_0(v)=\omega(W(v))$.

We have the following

Theorem 6.6. *Let ω be any invariant state on the CCR C^*-algebra* $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ *such those defined in the beginning of Section 6, extended in the obvious manner to all of $\mathfrak{M}_\omega$.*

If ν satisfies (6.1), then both limits

$$\lim_{t\to\pm\infty}\omega\circ\alpha^P_t:=\omega^{\pm}$$

exist in the $\sigma(\mathfrak{M}^*_\omega,\mathfrak{M}_\omega)$–topology.

We have an explicit expression for any generator $W(\mathfrak{m})$ as

$$\omega^{\pm}(W(\mathfrak{m}))=\sum_{n=0}^{\infty}\int_{\mathbb{R}^m}a^{\pm}_n(h(x))\mu(dx)=\int_{\mathbb{R}^m}\omega^{\pm}(W(h(x)))\mu(dx),$$

where $a^{\pm}_n$ is defined in (6.5).

Moreover, if ω is mixing w.r.t. the linear dynamics, then we have, again in the $\sigma(\mathfrak{M}^*_\omega,\mathfrak{M}_\omega)$–topology,

$$\lim_{t\to\pm\infty}\varphi\circ\alpha^P_t=\omega^{\pm}$$

for each state $\varphi \in \pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''_*$.

Proof. As the unit ball of $\mathfrak{M}^*_\omega$ is compact in the weak*-topology, it is enough to show that the assertion holds for any generator $W(\mathfrak{m})$.

Starting from the series (5.6) and exchanging the summation with integration we get, by Fubini Theorem,

$$\begin{aligned}&\omega(\alpha^P_t W(\mathfrak{m}))\\ &=\lim_{t\to\pm\infty}\sum_{n=0}^{+\infty}(\pm 1)^n\int_{\Delta^n_t\times\mathbb{R}^{m+n}}\omega(W(\hat{w}_n(h(x),\underline{t}^n,\underline{\lambda}^n))\hat{G}_n(\hat{w}_n(h(x),\underline{t}^n,\underline{\lambda}^n))\end{aligned}$$

where $\Delta^n_t=\{(t_1,\dots,t_n)\in\mathbb{R}^n\mid \pm t\geq \pm t_1\geq\cdots\geq\pm t_n\geq 0\}$. Next, letting $n\in\mathbb{N}$ be sufficiently large and choosing $s\in(\frac{2}{3},1)$ according to Lemma B.3, we have, by Lemma C.1,

$$\begin{aligned}\int_{\Delta^n_t\times\mathbb{R}^{m+n}}&\left|\omega(W(\hat{w}_n(h(x),\underline{t}^n,\underline{\lambda}^n))\hat{G}_n(\hat{w}_n(h(x),\underline{t}^n,\underline{\lambda}^n))\right|\\ &\leq(\ell\|\nu\|\|\gamma_{e_{0,1}}\|_1)^n\int_{\mathbb{R}^m}e^{c\|h(x)\|^s_{W^2(\mathbb{Z})^2}}|\mu|(dx)\\ &\leq C_\varepsilon(\ell\|\nu\|\|\gamma_{e_{0,1}}\|_1)^n\int_{\mathbb{R}^m}e^{\varepsilon\|h(x)\|_{W^2(\mathbb{Z})^2}}|\mu|(dx)\end{aligned}$$

where $\ell>2$ but $\ell\|\nu\|\|\gamma_{e_{0,1}}\|_1<1$.

The proof now follows easily by Lebesgue dominate convergence Theorem and, if ω is mixing w.r.t. the linear dynamics, taking into account (6.4). ☐

Corollary 6.7. *The states* $\omega^\pm$ *are invariant w.r.t. the evolution* α^P_t *and are regular when restricted to the CCR* C^**-algebra* $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$.

Proof. The invariance is immediate by the above result since $\omega^\pm(\alpha^P_t A)=\lim_{s\to\pm\infty}\omega(\alpha^P_{t+s}A)=\omega^\pm(A)$.

The regularity follows for the series defining $\omega^\pm(W(v))$ converges uniformly on bounded sets of $W^2_{\mathbb{R}}(\mathbb{Z})^2$ and $a^\pm_n(\lambda v)$ are continuous in λ by Lebesgue dominated convergence Theorem. ☐

It remains to investigate the unicity of the limiting state. The argument we use here may seem a bit indirect, yet it is the only available given that, in general, we do not know if $\omega^\pm$ are normal w.r.t. ω.

We begin our analysis with the following

Lemma 6.8. *For each* $v\in W^2_{\mathbb{R}}(\mathbb{Z})^2$ *we have*

$$\lim_{t\to\infty}\omega^\pm(W(T_t v))=\omega(W(v)).$$

Proof. Theorem 6.6 yields

$$\lim_{t\to\infty}\omega^{\pm}(W(T_tv)) = \lim_{t\to\infty}\sum_{n=0}^{\infty} a_n^{\pm}(T_tv)$$

where $a_n^{\pm}$ are defined in (6.5). Remembering Lemma C.1, we can exchange the limit with the integration. By Riemann-Lebesgue Lemma, we get $\lim_{t\to\infty} a_n^{\pm}(T_tv) = 0$ for each $n > 0$ and the result follows. □

The next result is the key to our approach to the identification of limiting states.

Lemma 6.9. *Let $\varphi \in \{\omega^-, \omega^+\}$. Then φ satisfies the following integral equation:*

$$\begin{aligned}\varphi(W(v)) =& \omega(W(v)) \\ &+ \int_0^{+\infty} ds \int_{\mathbb{R}} \nu(d\lambda)\varphi(W(\lambda e_{0,1} + T_sv))g(v,s,\lambda).\end{aligned} \tag{6.6}$$

Moreover, if ω is mixing w.r.t. the linear dynamics α_t^0 and $\varphi \in \pi_\omega(\overline{\mathfrak{A}(W_{\mathbb{R}}^2(\mathbb{Z})^2,\sigma)})''_*$ is any invariant state w.r.t. the non linear dynamics α_t^P, then φ satisfies also the integral equation (6.6).[12]

Proof. By Theorem 6.6 it follows

$$\begin{aligned}&\varphi(W(v)) = \varphi(\alpha_t^P W(v)) \\ =&\varphi(\alpha_t^0 W(v)) + \varphi\left(\int_0^t ds \int_{\mathbb{R}} \nu(d\lambda)\alpha_s^P(W\lambda e_{0,1} + T_{t-s}v)g(v,t-s,\lambda)\right) \\ =&\varphi(\alpha_t^0 W(v)) + \int_0^t ds \int_{\mathbb{R}} \nu(d\lambda)\varphi(W(\lambda e_{0,1} + T_{t-s}v))g(v,t-s,\lambda) \\ =&\varphi(\alpha_t^0 W(v)) + \int_0^t ds \int_{\mathbb{R}} \nu(d\lambda)\varphi(W(\lambda e_{0,1} + T_sv))g(v,s,\lambda)\end{aligned}$$

due to invariance of $\omega^{\pm}$ w.r.t. α_t^P. Taking the limit for $t \to +\infty$ in the above equation we obtain the assertion by Lemma 6.8.

Now, if φ is any invariant state in $\pi_\omega(\overline{\mathfrak{A}(W_{\mathbb{R}}^2(\mathbb{Z})^2,\sigma)})''_*$ we again obtain the assertion if ω is mixing w.r.t. α_t^0, see Remark 6.5. □

[12] The same computations as those contained in the proof show that, if φ is one of states under consideration, it satisfies also the other integral equation

$$\begin{aligned}\varphi(W(v)) =& \omega(W(v)) \\ &+ \int_0^{-\infty} ds \int_{\mathbb{R}} \nu(d\lambda)\varphi(W(\lambda e_{0,1} + T_sv))g(v,s,\lambda).\end{aligned}$$

The following theorem is the announced result on the equality of limit states.

Theorem 6.10. *On the C^*-algebra $\mathfrak{M}_\omega$ we have*

$$\omega^+ = \omega^- =: \omega_\infty .$$

Moreover, if there exists a state $\varphi \in \pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)})''_*$ invariant w.r.t. the time evolution α^P_t, then $\varphi = \omega_\infty$.

Proof. It is enough to verify the assertion for the (representative of the) Weyl operators.

We start considering the linear spaces $\mathcal{B}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$ ($\mathcal{B}_{\mathrm{loc}}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$) made of Borel bounded (locally bounded) complex valued functions together with the (not everywhere defined) linear map $F : \mathcal{B}_{\mathrm{loc}}(W^2_{\mathbb{R}}(\mathbb{Z})^2) \to \mathcal{B}_{\mathrm{loc}}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$ given by

$$F(f)(v) := \int_0^{+\infty} ds \int_{\mathbb{R}} \nu(d\lambda) f(\lambda e_{0,1} + T_s v) g(v, s, \lambda) .$$

It is immediate to show that F maps $\mathcal{B}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$ in $\mathcal{B}_{\mathrm{loc}}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$. In addition, from Lemma C.1 it follows that F^n is well defined on $\mathcal{B}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$. Indeed, we have the estimate

$$|F^n(f)(v)| \leq (2[\![\nu]\!] \|\gamma_{e_{0,1}}\|)^n \exp\left\{ \frac{\|\gamma_v\|_1}{2^s \|\gamma_{e_{0,1}}\|_1} \right\} c^{\frac{n \ln \ln n}{\ln n}} \|f\|_\infty \tag{6.7}$$

for each $f \in \mathcal{B}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$.

Accordingly, equation (6.6) can be written as

$$f = f_0 + F(f) \tag{6.8}$$

where $f_0(v) = \omega(W(v))$ and $f(v) = \varphi(W(v))$ since, in our situation, both functions belong to $\mathcal{B}(W^2_{\mathbb{R}}(\mathbb{Z})^2)$. Hence, we will analyze the bounded solutions of the fixed point equation (6.8) when f_0 is bounded.

If f is a solution of (6.8), then f satisfies also the following sequence of equations

$$f = \sum_{k=0}^{n} F^k(f_0) + F^{n+1}(f). \tag{6.9}$$

From the estimate (6.7), it follows that we can take the pointwise limit in (6.9) obtaining that each solution of (6.8) reads

$$f(v) = \sum_{k=0}^{\infty} F^k(f_0)(v) \tag{6.10}$$

uniformly on bounded sets of $W^2_{\mathbb{R}}(\mathbb{Z})^2$. On the other hand it is trivial to see that (6.10) is a solution of (6.8). Thus equation (6.8) has the unique solution given by (6.10). $\square$

The following theorem is one of the main results of this section.

Theorem 6.11. *Let the QDS* $(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)},\alpha^0_t,\omega)$ *be mixing. Then the QDS* $(\mathfrak{M}_\omega,\alpha^P_t,\omega_\infty)$ *is mixing as well.*

Proof. Clearly it suffices to check the mixing on the Weyl operators.

We start by noticing that $\omega_\infty(W(u)\alpha^P_tW(v))$ is given by an absolutely summable double series whose summands can be chosen indifferently (Theorem 6.10) as

$$
\begin{aligned}
a^\pm_{m,n}(t)=&\int_{\Delta_{\pm\infty}\times\mathbb{R}^m}\int_{\Delta_t\times\mathbb{R}^n}\\
&\times\omega(W(\hat w_m(u,\underline{t}^m,\underline{\lambda}^m))W(T_t\hat w_n(v,\underline{\tilde t}^n,\underline{\tilde\lambda}^n)))\\
&\times\hat G_m(u,\underline{t}^m,\underline{\lambda}^m)\hat G_n(v,\underline{\tilde t}^n,\underline{\tilde\lambda}^n)\,.
\end{aligned}
$$

Now we apply Lebesgue dominated convergence Theorem and obtain

$$
\begin{aligned}
\lim_{t\to\pm\infty}a^\pm_{m,n}(t)=&\int_{\Delta_{\pm\infty}\times\mathbb{R}^m}\int_{\Delta_{\pm\infty}\times\mathbb{R}^n}\\
&\times\omega(W(\hat w_m(u,\underline{t}^m,\underline{\lambda}^m)))\omega(W(\hat w_n(v,\underline{\tilde t}^n,\underline{\tilde\lambda}^n)))\\
&\times\hat G_m(u,\underline{t}^m,\underline{\lambda}^m)\hat G_n(v,\underline{\tilde t}^n,\underline{\tilde\lambda}^n)\\
&\equiv a^\pm_m(u)a^\pm_n(v)
\end{aligned}
$$

taking into account that

$$
\begin{aligned}
&\lim_{t\to\pm\infty}\omega(W(\hat w_m(u,\underline{t}^m,\underline{\lambda}^m))W(T_t\hat w_n(v,\underline{\tilde t}^n,\underline{\tilde\lambda}^n)))\\
&\quad=\omega(W(\hat w_m(u,\underline{t}^m,\underline{\lambda}^m)))\omega(W(\hat w_n(v,\underline{\tilde t}^n,\underline{\tilde\lambda}^n)))
\end{aligned}
$$

pointwise, due to the mixing property of ω.

Finally,

$$\begin{aligned}\lim_{t\to\pm\infty}\omega_\infty(W(u)\alpha^P_tW(v)) &= \lim_{t\to\pm\infty}\sum_{m,n=0}^{+\infty}a^\pm_{m,n}(t)\\ &= \sum_{m,n=0}^{+\infty}\lim_{t\to\pm\infty}a^\pm_{m,n}(t)\\ &= \sum_{m,n=0}^{+\infty}a^\pm_m(u)a^\pm_n(v)\\ &\equiv \omega_\infty(W(u))\omega_\infty(W(v))\,.\end{aligned}$$

□

We conclude our analysis on the long time behavior of quantum systems arising by nonlinear perturbations of the dynamics with some considerations about relative normality between ω and ω_∞ both regarded as states on $\mathfrak{M}_\omega$. In general we cannot give any answer to this interesting problem. Conversely, for KMS states, we have a full understanding of the situation, at least for sufficiently small perturbations. We would like to remark that a time-dependent perturbation theory for KMS states seems to be absent in the literature relatively to the case under consideration, as the dynamical system $(\mathfrak{M}_\omega,\alpha^P_t)$ is never a C^*-dynamical system and the mixing property (w.r.t. the unperturbed dynamics) for the states ω considered here seems to be not satisfied for all observables in the von Neumann algebra $\pi_\omega(\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)})''$, see below.

We recall the definition of mutually normality according to [31], Cap. III.

Let ω_i, $i=1,2$ be two states of a C^*-algebra $\mathfrak{A}$ with GNS representations π_i, $i=1,2$ respectively. Then ω_2 is *normal* w.r.t. ω_1 if there exists a normal homomorphism ρ of $\pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A})''$ such that

$$\rho\circ\pi_1=\pi_2.$$

In the abelian case, this definition corresponds to the fact that the Borel measure μ_2 on the spectrum of $\mathfrak{A}$, which describe the state ω_2, is absolutely continuous w.r.t the measure μ_1, relative to the state ω_1.

We have the following

Proposition 6.12. *The limit state ω_∞, when restricted to the CCR C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$, is locally normal w.r.t. the local structure determined by all finite dimensional sympletic subspaces of $W^2_{\mathbb{R}}(\mathbb{Z})^2$.*

Proof. This is nothing but the von Neumann uniqueness Theorem, see e.g. [7] Corollary 5.2.15, provided that the state ω_∞, when restricted to the CCR C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$, is regular. But the last assertion is contained in Corollary 6.7. □

6.4. **The convergence to the equilibrium for KMS states.**

In the case of KMS states, it is possible to have a more explicit description of the state, thanks to the time independent perturbation theory [7]. To explain these facts we fix our attention on the two W^*-dynamical systems (M,α^0_t), (M,α^P_t) where $M=\pi_{\omega_\beta}(\mathfrak{A}(W^2(\mathbb{Z})^2,\sigma))''\equiv\mathfrak{M}''_\omega$; ω_β is one of the β-KMS state introduced in Section 4.4.

Let Ω^P_β be the vector given in [7] Corollary 5.4.5 relative to Ω_β $(\omega_\beta(A):=\langle\Omega_\beta,\pi_{\omega_\beta}(A)\Omega_\beta\rangle)$.

We start with the following

Proposition 6.13. *If the QDS $(\overline{\mathfrak{A}(W^2(\mathbb{Z})^2,\sigma)}$, α^0_t, $\omega_\beta)$ is mixing, then*

$$\omega^P_\beta(A):=\langle\Omega^P_\beta,\pi_{\omega_\beta}(A)\Omega^P_\beta\rangle$$

is the unique β-KMS state of $\mathfrak{M}_\omega$ in the sector determined by the defining representation of $\mathfrak{M}_\omega$. It is a faithful factor state.

Proof. As ω is mixing we can conclude, by the mean ergodic Theorem, that Ω_β is the unique (up to a multiplicative constant) vector which is invariant for the unitary group e^{iHt} which implements the linear dynamics, see [18], Theorem 2.1. Then $M\wedge M'=\mathbb{C}I$ by [7] Proposition 5.3.29. So we again conclude that ω^P_β is the unique KMS normal state on M for the perturbed dynamics α^P_t. Moreover ω^P_β is faithful ([7] Theorem 5.3.10). □

We collect the results about the KMS states in the following

Theorem 6.14. *Let ω_β be any KMS state at inverse temperature β for the QDS $(\overline{\mathfrak{A}(W^2(\mathbb{Z})^2,\sigma)}$, α^0_t, $\omega_\beta)$ among those analyzed in the sequel and considered as a (vector) state on all of $\mathfrak{M}_{\omega_\beta}$.*

Then,

$$\lim_{t\to\pm\infty}\omega_\beta\circ\alpha^P_t=\omega^P_\beta$$

where ω^P_β is precisely the state given in [7], Corollary 5.4.5.

Moreover, ω^P_β is mixing on $\mathfrak{M}_{\omega_\beta}$ w.r.t the non linear dynamics α^P_t.

Proof. The proof consists of the previous results (Theorem 6.10 together with Theorem 6.11) specialized to the particular cases of KMS states. □

Another situation which is covered by our analysis is the time dependent theory for ground states relative to non linear perturbation of the dynamics.

Namely, relatively to ground states ω_∞ introduced in Proposition 4.8, we have that $\omega_\infty \circ \alpha_t^0$ converges , as $t \to \pm\infty$, to a unique state $\omega_{\infty,\infty}$. Unfortunately, it is unclear if $\omega_{\infty,\infty}$ is also a ground state on $\mathfrak{M}_\omega$ for the perturbed dynamics.

Let us conclude the section with few open questions. A natural question arises if one considers mixing β-KMS states together with nonlinear perturbations

$$P := \int_{\mathbb{R}} \pi_{\omega_\beta}(W(\lambda e_{0,1}))\nu(d\lambda)$$

with conditions only on $|\nu|(\mathbb{R})$ and no conditions on higher momenta of the measure ν. In this case, in order to conclude that

$$\lim_{t\to\pm\infty} \omega_\beta \circ \alpha_t^P = \omega_\beta^P,$$

it could seem natural to proceed in analogy with the arguments contained in [7], Proposition 5.4.13. Unfortunately, in order to argue along such lines, a stronger condition on clustering is required, that is strong clustering on all elements of the associated W^*-algebra, which seems not to be available in the present setting. More in general it would be very interesting to develop a non-perturbative approach to the questions treated in this paper.

Last we would like to remark that, if one considers quantum harmonic crystals in higher dimensions, then non mixing quasi free KMS could arise as a Bose condensation of phonon could take place, see [7], Section 5.2. In this case we could not conclude that the limiting state is a KMS state relative to the non linear dynamics.

Appendix A.

In this appendix we perform some necessary but standard and boring computations.

We start by considering, for $z \notin [-1, 0]$ and $k \in \mathbb{Z}$, the following functions

$$f_k(z) := \frac{1}{2}\int_{-1}^{1} \frac{e^{ik\pi x}}{\omega(x) - z} dx \tag{A.1}$$

where $\omega(x) = \frac{\cos \pi x - 1}{2}$. We note that f_k is even w.r.t. k. In order to compute the above sequences of functions, it is necessary to study the zeros of the polynomial

$$P(\zeta) := \zeta^2 - 2(1 + 2z)\zeta + 1$$

as we will see below. $P(\zeta)$ is equal zero for

$$\zeta_\pm = e^{\pm\eta}$$

where $2(1 + 2z) := e^\eta + e^{-\eta}$; $\eta = \alpha + i\beta$ with $\alpha \geq 0$ and $\beta \in [0, 2\pi)$. Or, in alternative,

$$\zeta_\pm = 1 + 2z \pm \sqrt{(1 + 2z)^2 - 1}. \tag{A.2}$$

where the square root is defined with the cut $[-1, 0]$ and with the sign chosen according to the preceding definition (that is, the root is positive for $z \in \mathbb{R}_+$). It follows that $|\zeta_\pm| = 1$ if and only if $\alpha = 0$, but this implies $\mathrm{Im}(1+2z) = 0$ and $|\mathrm{Re}(1+2z)| \leq 1$. That is $z \in [-1, 0]$, which is excluded by hypothesis. Hence, ζ_- will always lay inside the unit circle.

Now we are ready to prove the following

Lemma A.1. *For $z \notin [-1, 0]$ and $k \in \mathbb{Z}$ we have*

$$f_k(z) = -\frac{2\zeta_-(z)^{|k|}}{\sqrt{(1 + 2z)^2 - 1}}.$$

Proof. As f_k is even in k, we restrict ourselves to $k \geq 0$. After an elementary change of variable we get

$$f_k(z) = \frac{2}{\pi i}\int_{|\zeta|=1} \frac{\zeta^k}{\zeta^2 - 2(1 + 2z)\zeta + 1} d\zeta$$

where the unit circle is counterclockwise oriented. Now the denominator can be written as

$$P(\zeta, z) = (\zeta - \zeta_+(z))(\zeta - \zeta_-(z)).$$

So, by the above analysis, the integrand has only a simple pole $\zeta = \zeta_-(z)$ inside the unit circle. The proof now follows by the Residue Formula. $\square$

Taking into account

$$\frac{1}{2}\int_{-1}^{1}\frac{e^{ik\pi x}}{\omega(x)-z}\omega(x)dx=\delta_{k,0}+zf_k(z), \tag{A.3}$$

and

$$\frac{1}{2}\int_{-1}^{1}\frac{\rho+a\omega(x)}{\omega(x)-z}dx=a-\frac{2(\rho+az)}{\sqrt{(1+2z)^2-1}},$$

it follows that the function $\delta(z)$, defined in (3.5), reads as (3.7). It has a pole if

$$1-a-4\frac{\rho+az}{\zeta_- -\zeta_+}=0$$

that is

$$(1-a)(\zeta_- -\zeta_+)=4\rho-2a+a(\zeta_- +\zeta_+), \tag{A.4}$$

which implies

$$(1-2a)\zeta_- -\zeta_+=2(2\rho-a).$$

Since $\zeta_\pm=e^{\pm(\alpha+i\beta)}$, remembering that $a=1-\frac{1}{M}<1$, it is easy to see that the imaginary part of the above equation is satisfied only if $\sin\beta=0$. This implies that both $\zeta_\pm$, and z, must be real; in addition $\cos\beta\in\{-1,1\}$.

If $\cos\beta=1$, then $\zeta_\pm=e^{\pm\alpha}$ and

$$1+2z\equiv\cosh\alpha\geq 1$$

implies $z\geq 0$. Accordingly, (A.4) reads

$$\sinh\alpha=-2\frac{\rho+az}{1-a}\,. \tag{A.5}$$

Since, for $z\geq 0$, we have $\sinh\alpha=\sqrt{4z^2+4z}\geq 2z$, if $M<1$, hence $\rho\geq 0$, the r.h.s. of (A.5) reads

$$-\frac{2\rho}{1-a}+2(1-M)z<2(1-M)z<2z,$$

so (A.5) has a no solutions. If, on the other hand $M\geq 1$, equation (A.5) has a solution iff $\rho\leq 0$, that is iff $M\geq 1-\frac{K}{\mu}$.

If $\cos\beta=-1$, then $\zeta_\pm=-e^{\pm\alpha}$ and

$$-(1+2z)\equiv\cosh\alpha\geq 1$$

implies $z\leq -1$. Using again (A.4), we have

$$\sinh\alpha=2\frac{\rho+az}{1-a}. \tag{A.6}$$

The above equation has always a solution if $M < 1$, while if $M \geq 1$, equation (A.6) has a solution iff

$$M \leq 1 + \frac{K}{1-\mu}.$$

In other words $\delta(z)$ has no poles outside the interval $[-1, 0]$ iff $1 + \frac{K}{1-\mu} < M < 1 - \frac{K}{\mu}$; we will always assume such an inequality in the rest of this appendix.

Now we compute, according to the preceding lemma, all the matrix elements which we will need in the sequel. Namely, setting

$$e_k(x) := e^{i\pi kx},$$

we define ($R(z) := R_\Theta(z)$, see (3.6))

$$R_{kl}(z) := \langle e_k, R(z)e_l \rangle.$$

The following contour integrals

$$\frac{1}{2\pi i}\int_{\gamma_c} F(z)R_{kl}(z)dz$$

allow us to compute all the matrix elements of the operator $F(\Theta)$ when F is analytic in a neighborhood of $[-1, 0]$.

Lemma A.2. *For $z \in \mathbb{C}\backslash[-1, 0]$, we have*

$$R_{kl}(z) = f_{k-l}(z) - \delta(z)f_k(z)[(\rho + az)f_l(z) + a\delta_{l,0}].$$

Proof. Taking into account formulae (3.6) and (A.1) for $R(z)$ and f_k respectively, the assertion follows easily by (A.3). □

An easy but tedious calculation yields the next lemma.

Lemma A.3. *Let F be an analytic function on a neighborhood of $[-1, 0]$ and assume $1 + \frac{K}{1-\mu} < M < 1 - \frac{K}{\mu}$. Then*

$$-\frac{1}{2\pi i}\int_{\gamma_c} F(z)R_{kl}(z)dz = \frac{1}{\pi}\int_{-1}^{0} dxF(x)r_{kl}(x)$$

where $r_{kl}(x)$ is, for $x\in(-1,0)$, the continuous function given by

$$
\begin{aligned}
r_{kl}(x) :=& \frac{\cos[|k-l|b(x)]}{\sqrt{-x-x^2}}\\
&+\frac{1}{\sqrt{-x-x^2}[(\rho+ax)^2+(1-a)^2(-x-x^2)]}\\
&\times\{(a-a^2)(-x-x^2)\delta_{l,0}\cos[|k|b(x)]\\
&-(\rho+ax)^2\cos[(|k|+|l|)b(x)]\\
&+(a\rho+a^2x)\sqrt{-x-x^2}\delta_{l,0}\sin[|k|b(x)]\\
&+(1-a)(\rho+ax)\sqrt{-x-x^2}\sin[(|k|+|l|)b(x)]\}\,.
\end{aligned}
$$

Here $\sqrt{-x-x^2}$ is the arithmetic square root of $-x-x^2\geq 0$ and $b(x):=\arccos(1+2x)$.

Proof. The computation can be easily made as the contour γ_c collapse on the interval $[-1,0]$. Indeed, as it is well known, the square root $\sqrt{(1+2z)^2-1}$ has limits as z tends to $x\in[-1,0]$ from above and from below on the complex plane. These limits are (according to the definition of the square root) $\pm 2i\sqrt{-x-x^2}$ respectively. Moreover, by (A.2), we have for $x\in(-1,0)$

$$
\lim_{\varepsilon\to 0^+}|\zeta_\pm(x\pm i\varepsilon)|=1
$$

so that the argument

$$
b(x):=\lim_{\varepsilon\to 0^+}\beta(x+i\varepsilon)
$$

coincide, on the upper side of the cut, with $\arccos(1+2x)$ again by (A.2). The proof now follows via a direct computation by using Lemma A.1 together with Lemma A.2, remembering the expression (3.7) of $\delta(z)$ and the discussion on the poles of $\delta(z)$. In fact, all the functions appearing as integrands in the r.h.s. of the formula are summable on $(-1,0)$. The integral of the functions appearing in the r.h.s. of the formula, taken on a small circle (in the complex plane) of radius ε centered on any of the ramification points 0 and -1, are of order $O(\sqrt{\varepsilon})$ as ε goes to zero. ◻

Remark A.4. *A direct inspection of $r_{kl}(x)$, shows that*

$$
r_{kl}(x)=\sqrt{-x-x^2}(|k|+|l|+1)^2\tilde r_{kl}(x)
$$

where $\tilde r_{kl}(x)$ are continuous equibounded function on $[-1,0]$.

Furthermore, setting $r_{kl}(x)=0$ for $x\notin[-1,\,0]$, $r_{kl}(x)$ can be extended to an absolutely continuous function on all the real line with derivative in $L^p(\mathbb{R})$ for each $1\leq p<2$.

Appendix B.

Following the computations contained in the previous appendix, we investigate some relevant properties related to (possibly unbounded) functional calculi for the operator Θ.

We begin with the investigation of the explicit expression of $e^{t\Lambda^*}$ which (with an abuse of notation) describes the linear dynamics on the CCR C^*-algebras $\overline{(\mathfrak{A}(W^n_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$.

By Proposition 3.1 and formula (3.4), setting $R(z):=R_\Theta(z)$, we get

$$e^{\Lambda^* t}=-\frac{1}{2\pi i}\int_{\gamma_c}dz\sum_{n=0}^{\infty}t^{2n}z^n\begin{pmatrix}\frac{1}{(2n)!}\Gamma^{-1}R(z)\Gamma & \frac{tz}{(2n+1)!}\Gamma^{-1}R(z)\\ \frac{t}{(2n+1)!}R(z)\Gamma & \frac{1}{(2n)!}R(z)\end{pmatrix},$$

where the integral is meant in the norm sense.

Thus, defining the entire functions (w.r.t. the variable z)

$$h_p(z,t):=\sum_{n=0}^{\infty}\frac{z^nt^{2n}}{(2n)!}\,;\quad h_d(z,t):=\sum_{n=0}^{\infty}\frac{z^nt^{2n+1}}{(2n+1)!},$$

we have the following representation for the one parameter group T^μ_t which implements the dynamics on the phase space:

$$T^\mu_t= \tag{B.1}$$
$$-\frac{1}{2\pi i}\int_{\gamma_c}dz\begin{pmatrix}h_p(z+\mu,t)\Gamma^{-1}R(z)\Gamma & (z+\mu)h_d(z+\mu,t)\Gamma^{-1}R(z)\\ h_d(z+\mu,t)R(z)\Gamma & h_p(z+\mu,t)R(z)\end{pmatrix}$$

for $\Theta_\mu\equiv\Theta+\mu I$.

Continuing our analysis, we note that, due to Lemma A.3, the functions

$$\sum_{k,l}\overline{u_k}v_lr_{kl}(x)$$

have the rôle of the density of spectral measures, at least for finitely supported sequences. Then, If u, v are sequences with finite support, we set

$$r_{u,v}(x):=\sum_{k,l}\overline{u_k}v_lr_{kl}(x).$$

The following estimations is crucial in the sequel.

Proposition B.1. *The following assertions hold.*

(i) *Let $u,v\in L^2(\mathbb{Z})$ and $\{u_n\}$, $\{v_n\}$ sequences made of elements with finite support which converge in $L^2(\mathbb{Z})$ to u, v respectively. Then*

$$r_{u,v}(x):=\lim_n r_{u_n,v_n}(x)$$

uniquely defines a function in $L^1(-1,0)$ *which satisfies the following estimate*

$$\frac{1}{\pi}\int_{-1}^{0}|r_{u,v}(x)|dx \leq C_1\|u\|_{L^2(\mathbb{Z})}\|v\|_{L^2(\mathbb{Z})}. \tag{B.2}$$

(ii) *If* $u, v \in W^2_{\mathbb{R}}(\mathbb{Z})$, *then* $\frac{r_{u,v}(x)}{\sqrt{-x}}$ *belongs to* $L^\infty(-1,0)$ *and satisfies*

$$\sup_{x\in(-1,0)}\frac{|r_{u,v}(x)|}{\sqrt{-x}} \leq C_2\|u\|_{W^2(\mathbb{Z})}\|v\|_{W^2(\mathbb{Z})} \tag{B.3}$$

where sup *is the essential supremum on* $(-1,0)$.

Proof. (i) If u, v are elements of $L^2(\mathbb{Z})$ with finite support, then $r_{u,v}(x)$ is a continuous function on $(-1,0)$ (see Remark A.4). Now, taking into account that $\Gamma^{-\frac{1}{2}}F(\Theta)\Gamma^{\frac{1}{2}}$ is normal for each F analytic in a neighborhood of $(-1,0)$, we have by Proposition 3.1 and Lemma A.3,

$$\begin{aligned}\left|\frac{1}{\pi}\int_{-1}^{0}dxF(x)\sum_{k,l}\overline{u_k}v_l r_{kl}(x)\right| &= |\langle\Gamma^{\frac{1}{2}}u, \Gamma^{-\frac{1}{2}}F(\Theta)\Gamma^{\frac{1}{2}}\,\Gamma^{-\frac{1}{2}}v\rangle| \\ &\leq \|\Gamma^{\frac{1}{2}}u\|_2\|\Gamma^{-\frac{1}{2}}v\|_2\mathrm{spr}(\Gamma^{-\frac{1}{2}}F(\Theta)\Gamma^{\frac{1}{2}}) \\ &\leq \frac{1}{\sqrt{1-a}}\|u\|_2\|v\|_2\max_{x\in[-1,0]}|F(x)|\end{aligned}$$

where the last inequality follows by the Spectral Mapping Theorem and the fact that $\|\Gamma^{-\frac{1}{2}}\| = \frac{1}{\sqrt{1-a}}$. Now, as the analytic functions are dense in the space of all continuous functions on $[-1,0]$, we have (B.2) for finite supported sequences, with $C_1 := \frac{1}{\sqrt{1-a}}$.

Finally, if $u, v \in L^2(\mathbb{Z})$, we choose sequences $\{u_n\}$, $\{v_n\}$ made of elements with finite support which converge in $L^2(\mathbb{Z})$ to u, v respectively, we get

$$\begin{aligned}\int_{-1}^{0}|r_{u_n,v_n}(x) - r_{u_m,v_m}(x)|dx &\leq \int_{-1}^{0}|r_{u_n-u_m,v_n}(x)|dx \\ &+ \int_{-1}^{0}|r_{u_m,v_n-v_m}(x)|dx;\end{aligned}$$

that is, taking into account the above estimation, the sequence $\{r_{u_n,v_n}\}$ is a Chauchy sequence in $L^1(-1,0)$ which converges to a function $r_{u,v}$ in $L^1(-1,0)$. At the same time one can prove that this function does not depend on the sequences $\{u_n\}$, $\{v_n\}$ chosen in order to approximate u, v in $L^2(\mathbb{Z})$. An elementary application of Fatou Lemma yields that estimate (B.2) holds for general elements u, v of $L^2(\mathbb{Z})$.

(ii) Considering again finite supported sequences we have, after a Taylor expansion near 0^- of the circular functions appearing in the definition of $r_{kl}(x)$ (see Lemma A.3 and Remark A.4),

$$\frac{|r_{u,v}(x)|}{\sqrt{-x}} \leq C_2 \sum_k (k^2+1)|u_k| \sum_l (l^2+1)|v_l|.$$

The assertion follows immediately by the last estimate and a simple approximation argument in $W^2(\mathbb{Z})$. $\square$

For general elements u, v of $L^2(\mathbb{Z})$, we symbolically write for $r_{u,v}(x)$

$$r_{u,v}(x) = \sum_{k,l} \overline{u_k} v_l r_{kl}(x)$$

where the last formula is understood as a limit in $L^1(-1,0)$.

As the operator $\Gamma^{-\frac{1}{2}}\Theta\Gamma^{\frac{1}{2}}$ is self-adjoint (see Proposition 3.1), one can define the Borelian (possibly unbounded)[13] functional calculus for Θ. Then, if F is any Borel function on $\mathbb{R}_-$, we set

$$F(\Theta) := \Gamma^{\frac{1}{2}} F(\Gamma^{-\frac{1}{2}}\Theta\Gamma^{\frac{1}{2}})\Gamma^{-\frac{1}{2}}$$

where the domain of $F(\Theta)$ is determined by the domain of $F(\Gamma^{-\frac{1}{2}}\Theta\Gamma^{\frac{1}{2}})$, see [9, Part II]. Taking into account the above considerations, the next proposition follows by Proposition B.1 and Lemma A.3.

Proposition B.2. *The following assertions hold.*

(i) *If $F \in L^\infty(\mathbb{R})$ and v, $w \in L^2(\mathbb{Z})$, then*

$$\langle v,\, F(\Theta)w\rangle = \frac{1}{\pi}\int_{-1}^{0} dx F(x) r_{v,w}(x).$$

(ii) *If $F \in L^1(\mathbb{R}_-,\, \sqrt{-x}dx)$ and $v, w \in W^2(\mathbb{Z})$, then the bilinear form $Q_F(v,w)$ is well defined by*

$$Q_F(v,w) := \frac{1}{\pi}\int_{-1}^{0} dx F(x) r_{v,w}(x).$$

In addition, it is continuous on $W^2(\mathbb{Z}) \times W^2(\mathbb{Z})$.

---

[13] In the spirit of the following proposition, if $F \in L^2(\mathbb{R}_-,\, \sqrt{-x}dx)$ and $v, w \in W^2_{\mathbb{R}}(\mathbb{Z})$, then $v, w \in \mathrm{Dom}(F(\Theta))$ and

$$\langle \Gamma^{-1}F(\Theta)\Gamma v,\, F(\Theta)w\rangle = \frac{1}{\pi}\int_{-1}^{0} dx |F(x)|^2 r_{v,w}(x).$$

Then the bilinear form $Q_{|F|^2}$ of Proposition B.2 is given by

$$Q_{|F|^2}(v,w) = \langle \Gamma^{-1}F(\Theta)\Gamma v,\, F(\Theta)w\rangle.$$

Proof. The only non-trivial part is (ii). The difficulties is that the above mentioned unbounded functional calculus is clearly well defined only for $F \in L^2(\mathbb{R}_-, \sqrt{-x}dx)$, otherwise the domain of $F(\Theta)$ could be too small. Nevertheless, we are interested just in quadratic forms with domain containing $W^2(\mathbb{Z})$ which can be easily defined. Let $F_n \in L^\infty(\mathbb{R}_-, \sqrt{-x}dx)$ be a sequence converging to F in $L^1(\mathbb{R}_-, \sqrt{-x}dx)$. Then, for all $v, w \in W^2(\mathbb{Z})$

$$|\langle v, (F_n - F_m)w\rangle| \leq \frac{1}{\pi}\int_{-1}^{0} \frac{dx}{\sqrt{-x}}|F_n(x) - F_m(x)|\|v\|_{W^2(\mathbb{Z})}\|w\|_{W^2(\mathbb{Z})}.$$

Thus, we have a Chauchy sequence and the natural definition

$$Q_F(v, w) = \lim_{n\to\infty}\langle v, F_n w\rangle.$$

Clearly, the limit does not depend on the approximating sequence and enjoys the representation (ii). The continuity is straightforward. $\square$

One of the most important consequences of the previous result consists in the possibility of computing relevant dynamical functions as in the following

Lemma B.3. *For each $\mu \leq 0$, $\frac{2}{3} < s \leq 1$ and elements $v \in W^1(\mathbb{Z})^2$, the s-powers of the functions*

$$\gamma_v^\mu(t) := \langle v, e^{A_\mu t}e_{0,2}\rangle$$

are summable on the real line; furthermore

$$\lim_{t\to\infty}\gamma_v^\mu(t) = 0.$$

In addition, we have

$$\||\gamma_v^\mu|^s\|_1 \leq c_s(\mu)\|v\|^s_{W^1(\mathbb{Z})^2},$$
$$\|\gamma_v^\mu\|_\infty \leq c_\infty(\mu)\|v\|_{W^1(\mathbb{Z})^2},$$

Proof. Recalling the definition of S in (4.1), the representation (B.1) of T_t^μ yields

$$\begin{aligned}\gamma_v^\mu(t) = &\frac{2}{\pi}\int_{\sqrt{-\mu}}^{\sqrt{1-\mu}} d\xi \sin(\xi t)\sum_{k\in\mathbb{Z}}(Sv)_k^1 r_{k0}(-(\xi^2+\mu))\\ &+\frac{2}{\pi}\int_{\sqrt{-\mu}}^{\sqrt{1-\mu}} d\xi\xi \cos(\xi t)\sum_{k\in\mathbb{Z}}(Sv)_k^2 r_{k0}(-(\xi^2+\mu));\end{aligned}$$

where

$$\begin{aligned} r_{k0}(x) =& \{(1-a)^2(-x-x^2)+(\rho+ax)^2\}^{-1} \\ & \times \left\{(1-a)\sqrt{-x-x^2}\cos(b(x)|k|)+(\rho+ax)\sin(b(x)|k|)\right\}. \end{aligned}$$

Since $r_{k0}(0) = r_{k0}(-1) = 0$, the above functions can be extended to a continuous function on $\mathbb{R}$ by setting them equal to zero outside $[-1, 0]$. Riemann-Lebesgue Lemma together with Proposition B.1 yields that $\gamma_v^\mu(t)$ is a continuous function which vanishes at infinity so we have the second estimate for some constant $c_\infty(\mu)$.

Next, for $\xi \in [\sqrt{-\mu}, \sqrt{1-\mu}]$, one easily recovers

$$\left|\frac{d}{d\xi} r_{k0}(-(\xi^2+\mu))\right| \leq c_1|k|[(\xi^2+\mu)-(\xi^2+\mu)^2]^{-\frac{1}{2}}. \tag{B.4}$$

Clearly, for a finite supported element $w \in W^1(\mathbb{Z})$, r_{w,e_0} belongs to $L^p(\mathbb{R})$ for each $1 \leq p < 2$. Furthermore, we compute

$$\int_{\sqrt{-\mu}}^{\sqrt{1-\mu}} d\xi r_{w,e_0}(-(\xi^2+\mu))\sin\xi t = \frac{1}{t}\int_{\mathbb{R}} d\xi \left[\frac{d}{d\xi} r_{w,e_0}(-(\xi^2+\mu))\right]\cos\xi t\,.$$

Accordingly, if we set $g(\xi) = -ir_{(Sv)^1,e_0} + \xi r_{(Sv)^2,e_0}$, so that $\gamma_v = \frac{2\sqrt{2}}{\sqrt{\pi}}\mathrm{Re}[F(g)]$ (where $F(\cdot)$ stands for the Fourier transform),

$$\int_{\mathbb{R}} |\gamma_v(t)|^s dt \leq c_2\int_{\mathbb{R}} \frac{|F(g)|^s}{|t|^s+1}dt + c_2\int_{\mathbb{R}} \frac{|F(g')(t)|^s}{|t|^s+1}dt\,.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} |\gamma_v(t)|^s dt \leq & c_2\left[\int_{\mathbb{R}} (|t|^s+1)^{-(1-\frac{s}{2})^{-1}}\right]^{1-\frac{s}{2}} \|g\|_2^s \\ & + c_3\left[\int_{\mathbb{R}} (|t|^s+1)^{\frac{-3}{2}}\right]^{\frac{2}{3}} \|F(g')\|_{3s}^s \\ \leq & c_4\|g\|_\infty + c_5\|g'\|^s_{(1-\frac{1}{3s})^{-1}} \leq c_6\|v\|^s_{W^1(\mathbb{Z})^2} \end{aligned}$$

where we have used the Hölder and the Haulsdorff-Young inequalities since $\frac{3}{2} \leq (1-\frac{1}{3s})^{-1} < 2$.

By the above estimate we have the assertion for finitely supported sequences. Now, if $v \in W^1(\mathbb{Z})^2$, we choose a sequence $\{v_n\}$ made of elements with finite support which converges in $W^1(\mathbb{Z})^2$ to v. We have, by Fatou Lemma,

$$\begin{aligned} \int |\gamma_v^\mu(t)|^s dt \leq & \liminf \int |\gamma_{v_n}^\mu(t)|^s dt \\ \leq & c_s(\mu)\|v\|^s_{W^1(\mathbb{Z})^2} \end{aligned}$$

which is the assertion. □

Appendix C.

This appendix contains a crucial estimate for the study of the long time behavior of the non linear system.

We recall that $\Delta_t^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \ : \ t \geq t_1 \geq \dots \geq t_n \geq 0\}$, γ_w is defined in (5.4), $\hat{G}$ is given in (5.12) and finally $\|\nu\|$ is defined in (6.1).

Lemma C.1. *For each $s \in (\frac{2}{3}, 1]$ we have the estimate*

$$\int_{\Delta_t^n \times \mathbb{R}^n} |\hat{G}(v, \underline{t}^n, \underline{\lambda}^n)| dt_1 \dots dt_n |\nu|(d\lambda_1) \dots |\nu|(d\lambda_n)$$
$$\leq 2^n \|\nu\|^n \|\gamma_{e_{0,1}}\|_1^n \exp\left\{ \frac{\| |\gamma_v|^s \|_1}{2^s \|\gamma_{e_{0,1}}\|_1} \right\} c^{\frac{n \ln(\ln n)}{\ln n}}$$

where $c > e$ is some constant.

Proof. We start by setting

$$\gamma(t) := \gamma_{e_{0,1}}(t), \quad \Gamma(t) := \gamma_v(t)$$

as $e_{0,1}$ and v are kept fixed through the following analysis.

Using the representation (5.12), it follows

$$|\hat{G}(v, \underline{t}^n, \underline{\lambda}^n)| = 2^n \prod_{k=1}^n \left| \sin\left(\frac{\lambda_k}{2} \sum_{j=k+1}^n \lambda_j \gamma(t_k - t_j) + \Gamma(t_k) \right) \right|$$
$$\leq 2^n \prod_{k=1}^n \left[\left| \sin\left(\frac{\lambda_k}{2} \sum_{j=k+1}^n \lambda_j \gamma(t_k - t_j) \right) \right| + \left| \sin(\frac{\lambda_k}{2} \Gamma(t_k)) \right| \right]$$
$$\leq 2^n \prod_{k=1}^n \left[\sum_{j=k+1}^n \left| \frac{\lambda_k \lambda_j}{2} \right| |\gamma(t_k - t_j)| + \left| \sin(\frac{\lambda_k}{2} \Gamma(t_k)) \right| \right]$$
$$= \sum_{j=0}^{n-1} 2^{n-j} \sum_{\substack{A \subset \{1,\dots,n-1\} \\ \#A = j}} \prod_{k \notin A} \left| \sin(\frac{\lambda_k}{2} \Gamma(t_k)) \right| \prod_{k \in A} \left[\sum_{l=k+1}^n |\lambda_k \lambda_l| |\gamma(t_k - t_l)| \right]$$
$$= \sum_{j=0}^{n-1} 2^{n-j} \sum_{\substack{A \subset \{1,\dots,n-1\} \\ \#A = j}} \sum_{\sigma \in \Xi_A} \prod_{k \notin A} \left| \sin(\frac{\lambda_k}{2} \Gamma(t_k)) \right| \prod_{k \in A} |\lambda_k \lambda_{\sigma(k)}| |\gamma(t_k - t_{\sigma(k)})|,$$

where $\Xi_A := \{\sigma : A \to \{2, \dots, n\} \mid \sigma(k) > k\}$. Next, we would like to integrate the above expression. The difficulties come from the fact that the integrand does not have the invariance w.r.t. the permutations of the variables which would make easy to handle the domain of the integral. It is then convenient to "symmetrize" the above expression, nevertheless this must be done with some care in order not to spoil the subsequent estimates. To do this, we consider the following $n \times n$ matrices

$$[B(A,\sigma)]_{kl} := \begin{cases} \delta_{kl} & k \notin A \\ \delta_{kl} - \delta_{\sigma(k)l} & k \in A. \end{cases}$$

for each $\sigma \in \{1, \ldots, n\}^A$ with $|A| \leq n$, and define

$$\widetilde{\Xi}_A = \{\sigma \in \{1, \ldots, n\}^A \mid |\det B(A,\sigma)| = 1\}.$$

As $|\det B(A,\sigma)| = 1$ if $\sigma \in \Xi_A$, we have $\Xi_A \subset \widetilde{\Xi}_A$. Moreover a permutation $\pi \in \mathbb{P}_n$ acts in a natural way on $\sigma \in \{1, \ldots, n\}^A$ and it is easily seen that $\widetilde{\Xi}_A$ is globally invariant under this action.

Continuing our analysis we get

$$\begin{aligned}\int_{\Delta_t^n \times \mathbb{R}^n} |\hat{G}| \leq & \sum_{j=0}^{n-1} 2^{n-j} \sum_{\substack{A \subset \{1,\ldots,n\} \\ \#A = j}} \sum_{\sigma \in \widetilde{\Xi}_A} \int_{\Delta_t \times \mathbb{R}^n} \prod_{k \notin A} \left| \sin(\frac{\lambda_k}{2} \Gamma(t_k)) \right| \\ & \prod_{k \in A} |\lambda_k \lambda_{\sigma(k)}| |\gamma(t_k - t_{\sigma(k)})| \\ \leq & \frac{1}{n!} \sum_{j=0}^{n-1} 2^{n-j} \sum_{\substack{A \subset \{1,\ldots,n\} \\ \#A = j}} \sum_{\sigma \in \widetilde{\Xi}_A} \int_{\mathbb{R}^n \times \mathbb{R}^n} \prod_{k \notin A} \left| \sin(\frac{\lambda_k}{2} \Gamma(t_k)) \right| \\ & \prod_{k \in A} |\lambda_k \lambda_{\sigma(k)} \gamma(t_k - t_{\sigma(k)})|,\end{aligned}$$

where we have used the invariance by permutations.[14]

[14]Given any function $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^4 \to \mathbb{R}$ and, for each permutation π of $\{1, \ldots, n\}$ and $L > 0$, defining $\Delta(\pi) := \{\underline{t} \in \mathbb{R}^n \mid 0 \leq t_{\pi^{-1}(k)} \leq t_{\pi^{-1}(k+1)} \leq L\}$, it follows

$$\begin{aligned}& \sum_{\substack{A \subset \{1,\ldots,n\} \\ \#A = j}} \sum_{\sigma \in \widetilde{\Xi}_A} \int_{\Delta(\mathrm{id}) \times \mathbb{R}^n} \prod_{k \notin A} f(\lambda_k, t_k) \prod_{k \in A} g(\lambda_k, t_k, \lambda_{\sigma(k)}, t_{\sigma(k)}) \\ \leq & \frac{1}{n!} \sum_{\substack{A \subset \{1,\ldots,n\} \\ \#A = j}} \sum_{\sigma \in \widetilde{\Xi}_A} \int_{[0,L]^n \times \mathbb{R}^n} \prod_{k \notin A} f(\lambda_k, t_k) \prod_{k \in A} g(\lambda_k, t_k, \lambda_{\sigma(k)}, t_{\sigma(k)}).\end{aligned}$$

To see it it is enough to apply the change of variables $(\lambda'_k, t'_k) = (\lambda_{\pi(k)}, t_{\pi(k)})$ to the first member of the above expression whereby obtaining

$$\sum_{\substack{A \subset \{1,\ldots,n\} \\ \#A = j}} \sum_{\sigma \in \widetilde{\Xi}_A} \int_{\Delta(\pi) \times \mathbb{R}^n} \prod_{k \notin A} f(\lambda_k, t_k) \prod_{k \in A} g(\lambda_k, t_k, \lambda_{\sigma(k)}, t_{\sigma(k)})$$

Let us now consider the change of variable $(\lambda'_k, t'_k) = (\lambda_k, t_k)$ if $k \notin A$ and $(\lambda'_k, t'_k) = (\lambda_k, t_k - t_{\sigma(k)})$ if $k \in A$. The Jacobian of such a transformation is exactly $|\det B(A,\sigma)|$ which is precisely 1 by definition. Accordingly, the following estimate is obtained

$$\int_{\Delta_t^n \times \mathbb{R}^n} |\hat{G}| \leq \frac{1}{n!} \sum_{j=0}^{n-1} \binom{n}{j} 2^{n-j}$$
$$\times \sum_{\sigma \in \{1,\dots,n\}^j} \int_{\mathbb{R}^n \times \mathbb{R}^n} \prod_{k=1}^{j} |\lambda_k \lambda_{\sigma(k)} \gamma(t_k)| \prod_{k=j+1}^{n} \sin\left(\left|\frac{\lambda_k}{2}\Gamma(t_k)\right|\right).$$

To continue, let us notice that, for each $s \in (\frac{2}{3}, 1]$ and $\theta > 0$, we can write

$$\begin{aligned}\int_{\mathbb{R}} |\sin(\theta\Gamma(t))| &\leq \int_{\{|\Gamma(t)|>\theta^{-1}\}} 1 + \int_{\{|\Gamma(t)|\leq\theta^{-1}\}} |\sin(\theta\Gamma(t))| \\ &\leq \int_{\{|\Gamma(t)|>\theta^{-1}\}} \theta^s|\Gamma(t)|^s + \int_{\{|\Gamma(t)|\leq\theta^{-1}\}} \theta|\Gamma(t)|^{1-s}|\Gamma(t)|^s \\ &\leq \theta^s \||\Gamma|^s\|_1,\end{aligned}$$

where we have used the results of Lemma B.3. Moreover, notice that, for each $s \in (0, 1]$ and $k > 0$,

$$\begin{aligned}&\int_{\mathbb{R}} |\nu|(d\lambda)|\lambda|^{k+s} \leq \left[\frac{\int_{\mathbb{R}} |\nu|(d\lambda)|\lambda|^{k+1}}{\int_{\mathbb{R}} |\nu|(d\lambda)|\lambda|^{k}}\right]^s \int_{\mathbb{R}} |\nu|(d\lambda)|\lambda|^{k} \\ &\equiv \left[\frac{k!}{(k-1)!}\int_{\mathbb{R}} |\nu|(d\lambda)\frac{|\lambda|^{k+1}}{k!}\right]^s \left[\int_{\mathbb{R}} |\nu|(d\lambda)\frac{|\lambda|^{k}}{(k-1)!}\right]^{1-s} (k-1)! \\ &\leq k^s(k-1)!\llbracket\nu\rrbracket \leq k!\llbracket\nu\rrbracket\end{aligned}$$

where, keeping in mind the definition (6.1) of $\llbracket\nu\rrbracket$, we have used the property (5.3) for ν and Jensen inequality.

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and then notice that the $\Delta(\pi)$ are all disjoint and their union is exactly $[0, L]^n$.

Collecting the last results and using again a bit of combinatorics[15], we obtain

$$\int_{\Delta_t^n\times\mathbb{R}^n}|\hat{G}|\le\frac{\|\nu\|^n}{n!}\sum_{j=1}^n\binom{n}{j}^2 2^{(n-j)(1-s)}\sum_{l=1}^j\sum_{\substack{\{k_1,\dots,k_l\}\\ \sum_{m=1}^l k_m=j}}\|\gamma\|_1^j\||\Gamma|^s\|_1^{n-j}j!$$

$$\le\frac{\|\nu\|^n}{n!}\sum_{j=1}^n\binom{n}{j}^2 2^{(n-j)(1-s)}j!\|\gamma\|_1^j\||\Gamma|^s\|_1^{n-j}\sum_{l=1}^j\binom{j}{l-1}$$

$$\le 2^n\|\nu\|^n\|\gamma\|_1^n\sum_{j=1}^n\binom{n}{j}\frac{2^{-js}}{j!}\|\gamma\|_1^{-j}\||\Gamma|^s\|_1^j.$$

Since $\binom{n}{k}\le\frac{n^k}{k!}$ and $k!\ge k^ke^{-k}$, for each $x>0$ and $L<\frac{n}{2}$, it holds

$$\sum_{j=0}^n\binom{n}{j}\frac{x^j}{j!}\le\sum_{j=0}^{[L]-1}\binom{n}{[L]}\frac{x^j}{j!}+\sum_{j=[L]}^n 2^n\frac{x^j}{j!}$$

$$\le\sum_{j=0}^{[L]-1}\left(\frac{ne}{[L]}\right)^{[L]}\frac{x^j}{j!}+\frac{2^nx^{[L]}}{[L]!}\sum_{j=0}^{n-[L]}\frac{x^j}{j!}\frac{j![L]!}{(j+[L])!}$$

$$\le\left\{\left(\frac{ne}{L}\right)^L+\frac{2^nx^Le^L}{L^L}\right\}e^x.$$

So, by choosing $L=\frac{n}{\ln n+3}$ in the above estimate, we finally have,

$$\int_{\Delta_t^n\times\mathbb{R}^n}|\hat{G}|\le 2^n\|\gamma\|_1^n\|\nu\|^n e^{\frac{\||\Gamma|^s\|_1}{2^s\|\gamma\|_1}}c^{\frac{n\ln\ln n}{\ln n}}.$$

where c is a suitable constant greater than e. □

[15]Let $\{k_1,\dots,k_l\}$ be a partition of j (i.e. $\sum_{m=1}^l k_m=j$). The number of $\sigma\in\{1,\dots,n\}^j$ such that $\#\sigma^{-1}(i)\neq 0$ are exactly $\{k_1,\dots,k_l\}$, is precisely

$$\binom{n}{j}\binom{j}{k_1}\cdots\binom{j-k_1-\cdots-k_{l-1}}{k_l}=\binom{n}{j}[k_1!\cdots k_l!]^{-1}j!;$$

while all the possible partition in l elements are less than

$$\binom{j}{l-1}$$

which is the number of ways in which one can section the string $(1,\dots,j)$ in l segments (the size of each segment is the size of the one element of the partition).

For more details see [10].

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Francesco Fidaleo, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `fidaleo@mat.uniroma2.it`

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `liverani@mat.uniroma2.it`