Decay of Correlations in Hyperbolic Systems

CARLANGELO LIVERANI

Math.Dept., University of Rome *Tor Vergata*
Via della Ricerca Scientifica–00133 Rome, Italy
e-mail: Liverani@mat.utovrm.it

Abstract - I illustrate a unified approach to the study of the decay of correlations in hyperbolic dynamical systems.

The decay of correlations or, alternatively, the rate of approach of some initial distribution to an invariant one, is a widely studied problem in dynamical systems as well as in other fields.

Although a full understanding is not yet available, some notable results have been obtained in the case of hyperbolic dynamical systems. The main strategy in deriving such results is the study of the transfer operator (also called Perron-Frobenius, or Ruelle-Perron-Frobenius, operator), either directly (as in the case of some one-dimensional systems) or indirectly (after coding the dynamics by introducing special partitions of the phase space called Markov partitions).

I suggest a new approach to the study of the Perron-Frobenius operator, that avoids Markov partitions altogether, and permits a direct study of its properties also in the multi-dimensional case. Such a technique allows not only to reproduce, in a more direct way, most of the known results but also to obtain new ones (e.g., exponential decay of correlations for a wide class of discontinuous maps).

In essence, it is possible to construct metrics (Hilbert metrics) with respect to which the Perron-Frobenius operator is a contraction. Such a contraction allows one to obtain the invariant measure (if not already known) by an elementary, and constructive, fixed point theorem, and automatically implies an exponential rate for the decay of correlations.

The Hilbert metric is naturally defined on convex sets and is of interest in hyperbolic geometry (e.g., the disk equipped with the Hilbert metric is the Poincaré disk). If $\widetilde{\mathcal{C}}$ is a closed convex set in a topological space $\mathbb{V}$ then, given $x,\, y \in \widetilde{\mathcal{C}}$, one can define the line going thruough x and y and call $u,\, v$ the two intersections of such a line with $\partial\widetilde{\mathcal{C}}$ (set them equal to ∞ if there are no intersection). The Hilbert metric Θ in $\widetilde{\mathcal{C}}$ is defined as

$$\Theta(x,\, y) = \left| \ln \frac{|x-u||y-v|}{|x-v||y-u|} \right| .$$

That is, $\Theta(x,\, y)$ is the logarithm of the cross ratio of the points $x,\, y,\, u,\, v$.

Here we will consider convex cones in a linear vector space. Taking the quotient of the cone with respect to directions (the equivalence relation is: $f \sim g$ if there exists $\lambda \in \mathbb{R}^+$ such that $f = \lambda g$) we get a convex set that can be equipped with the Hilber metric. An explicit description of the semi-metric so obtained in the convex cone can be found in [L1]. For the present purposes all we need is the following theorem, originally due to Garrett Birkhoff [B] (the theorem is quite natural if one thinks about the invariant properties of the cross ratio).

Theorem 1.1. *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two vector spaces; $\widetilde{T} : \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $\widetilde{T}(\mathcal{C}_1) \subset \mathcal{C}_2$, for two closed convex cones $\mathcal{C}_1 \subset \mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{f,\, g \in \widetilde{T}(\mathcal{C}_1)} \Theta_2(f, g)$ we have*

$$\Theta_2(\widetilde{T}f, \widetilde{T}g) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta_1(f, g) \qquad \forall f, g \in \mathcal{C}_1$$

($\tanh(\infty) = 1$).

Remark 1.2. *If $\widetilde{T}(\mathcal{C}_1) \subset \mathcal{C}_2$ then it follows that $\Theta_2(\widetilde{T}f, \widetilde{T}g) \leq \Theta_1(f, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.*

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of $\widetilde{T}$ (see [B] for details).

Theorem 1.1 will be our basic tool in the following analysis. To see how to use it let us look at a simple example (a one dimensional expansive Markov map).

Let I_1, I_2 be two closed intervals such that $I_1 \cup I_2 = [0, 1]$, $I_1 \cap I_2 = \partial I_1 \cap \partial I_2$, and $T_i : I_i \to [0, 1]$, one to one and onto, such that $T_i \in C^{(2)}(I_i)$ and $DT_i \geq \lambda > 1$. We consider the map[1]

$$T(x) = \begin{cases} T_1(x) & \text{if } x \in I_1 \\ T_2(x) & \text{if } x \in I_2 \end{cases} \tag{1.1}$$

Let us see how the map can be lifted to the measures. Given $f, g \in L^2([0, 1])$

$$\int_0^1 f \circ T(x) g(x) dx = \int_0^1 f(x) \left[\sum_{i=1}^{2} g \circ T_i^{-1}(x) |D_x T_i^{-1}|\right] dx = \int_0^1 f\widetilde{T}g.$$

The operator $\widetilde{T}$, which describe how the density of the measure $g(x)dx$ is transformed under the iteration of the map T, is the Perron-Frobenius operator.

The result will follow from the study of the semigroup generated by the operator $\widetilde{T}$. One proves that $\widetilde{T}$ maps an appropriate cone of functions strictly inside itself, and therefore it has a fixed point that corresponds to the density of the unique absolutely continuous invariant measure.

Define the following cone

$$\mathcal{C}_a = \left\{ g \in C^{(0)} \;\middle|\; \forall x, y \in [0, 1] \;\; g(x) > 0;\; \frac{g(x)}{g(y)} \leq e^{a|x-y|} \right\}.$$

[1] In fact, the present technique applies to a much larger class of one dimensional maps (for example any expansive Markov map, and a wide class of non-Markov maps [L2]).

Lemma 1.3. *Given $\sigma \in (\lambda^{-1}, 1]$, $\widetilde{T}\mathcal{C}_a \subset \mathcal{C}_{\sigma a}$ provided $a \geq \frac{D}{\sigma - \lambda^{-1}}$, where*

$$D = \sup_{\substack{x \in [0, 1] \\ i \in \{1, 2\}}} \left| \frac{D_x^2 T_i^{-1}}{D_x T_i^{-1}} \right|.$$

In addition, using the Hilbert metric associated to $\mathcal{C}_a$,

$$\Delta = diam(\mathcal{C}_{\sigma a}) \leq 2 \ln \frac{1+\sigma}{1-\sigma} + 2\sigma a.$$

Lemma 1.3 is all we need to obtain:

Theorem 1.4 (Decay of correlations). *There exist $K, r \in \mathbb{R}^+$ such that, for each $f \in L^1([0, 1])$, $g \in C^{(1)}([0, 1])$ with $\int_0^1 g = 1$*

$$\left| \int_0^1 f \circ T^n g dx - \int_0^1 f \varphi_* dx \right| \leq K \|f\|_1 (\|g\|_1 + r\|g'\|_\infty) \Lambda^n$$

It is also possible to obtain explicit bounds for K, r and Λ; in fact, constructiveness is a general feature of the proposed approach. Next, let us see how these ideas can be used in a more general contest.

Consider, for example, a two dimensional compact smooth manifold $\mathcal{M}$, and an area preserving map $T : \mathcal{M} \to \mathcal{M}$. Assume the map T to be uniformly hyperbolic. Further, assume that the map enjoys the mixing property.

In the present case the Perron-Frobenius operator $\widetilde{T} : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ is given by

$$\widetilde{T}g = g \circ T^{-1}.$$

Due to the presence of both contracting and expanding directions the approach used for the one dimensional case must be slightly modified. To overcome this difficulty we will, in some sense, quotient away (or, better yet, "integrate away") the unwanted direction through the introduction of suitable test functions. This is achieved by introducing a class of closed, connected and differentiable curves

$$\Gamma_\delta = \{\gamma \subset \mathcal{M} \mid \delta \leq \text{length}(\gamma) \leq 2\delta;\ \forall x \in \gamma,\ \gamma'(x) \subset \mathcal{C}(x);\ |\kappa_\gamma(x)| \leq H\}$$

where $\gamma'(x)$ is the unitary tangent vector to γ at x, $\mathcal{C}(x)$ is a cone of directions (in the tangent space) close to the stable direction at x, $\kappa_\gamma(x)$ is the curvature of γ at x, and $\delta \in (0, 1)$. The idea is to perform integrals on γ, with respect to a large class of measures. We proceed to characterize such class by specifying the properties of their densities with respect to the measure induced by arc-length.

Let us define cones of "density" functions. Given any connected smooth curve γ, let

$$\widetilde{\mathcal{D}}_a(\gamma) = \left\{ f \in C^{(0)}(\gamma) \mid f > 0;\ \frac{f(x)}{f(y)} \leq e^{a d(x, y)^\nu} \right\}$$

where d means the arc-length distance along γ, and $\nu \in (0, \frac{1}{2}]$. In other words, we are considering positive functions whose logarithm has a bounded modulus of ν-Hölder continuity.

The reader will certainly notice the similarity with the cones used in the one dimensional case. Let us call $\rho_\gamma(f_1, f_2)$ the Hilbert metric in $\widetilde{\mathcal{D}}_a(\gamma)$.

It is now possible to introduce the cone of functions used in the present case:[2]

$$\mathcal{C}_{b,c} = \Bigg\{ g \in C^{(1)}(\mathcal{M}) \;\Bigg|\; \forall \gamma \in \Gamma_\delta,\ \forall x \in \gamma,\ \forall f,\ f_1,\ f_2 \in \widetilde{\mathcal{D}}_a(\gamma)\ \int_\gamma gf > 0;$$
$$\frac{f_2(x)\int_\gamma g f_1}{f_1(x)\int_\gamma g f_2} \le e^{b\rho_\gamma(f_1, f_2)};\ \|D^u g\|_\infty \le c \inf_{\substack{\gamma\in\Gamma_\delta \\ f\in\mathcal{D}_a(\gamma)}} \frac{\int_\gamma fg}{\int_\gamma f} \Bigg\}$$

where, by D^u we mean the derivative in the unstable direction.

As anticipated, once we integrate along the "problematic" direction, we can consider a cone quite similar to the previous ones. In fact, if we forget the actual value of g point by point and we consider only the values of the various averages, then the first condition in the cone is the usual positivity requirement; the second states that averages done over the same curve, with respect to similar measures,[3] are almost equal; and the third condition is very similar to a bound on the derivative of the logarithm of g (as in the one dimensional case).

Using the cone $\mathcal{C}_{b,c}$ it is not too difficult to obtain the equivalent of Lemma 2.3 also the multi-dimensional case, from which the equivalent of Theorem 2.4 (i.e., exponential decay of correlations) follows.

All we have seen, up to now, are known results and can be obtained by more traditional methods, but the present technique can be successfully applied also to new cases. For example, by using a more complex cone of functions (but essentially similar to the latter one) it is possible to consider discontinuous maps. To have in mind a concrete example, one can consider the following: $\mathcal{M} = \mathbb{T}^2$, $T(x, y) = F(A(x, y))$ where

$$A = \begin{pmatrix} 1 & a \\ a & 1 + a^2 \end{pmatrix}$$

with $a \notin \mathbb{N}$ (which implies that the linear part is discontinuous on the torus); T is area preserving; $F : \mathbb{T}^2 \to \mathbb{T}^2$ is any C^2 function sufficiently close to the identity so that the hyperbolicity is not spoiled.

Another interesting feature of the present method is the possibility to treat random perturbations of dynamical systems on the same footing as the unperturbed systems.

In fact, I belive that the full range of applicability of this approach has still to be clarified.

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References

[B] Garret Birkhoff, *Extension of Jentzsch's Theorem*, Transaction American Mathematical Society **85** (1957), 219–227.

[L1] C. Liverani, *Decay of correlations*, to appear in Annals of Mathematics.

[L2] C. Liverani, *Decay of correlations in piecewise expanding maps*, preprint.

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[2] The integration is done with respect to arc-length.

[3] Two measure are similar if the distance of their densities, in the Hilbert metric, is small.