

RANDOM WALK IN DETERMINISTICALLY CHANGING ENVIRONMENT

DMITRY DOLGOPYAT AND CARLANGLO LIVERANI

ABSTRACT. We consider a random walk with transition probabilities weakly dependent on an environment with a deterministic, but strongly chaotic, evolution. We prove that for almost all initial conditions of the environment the walk satisfies the CLT.

1. INTRODUCTION

The continuing interest in the limit properties of random walks has generated a remarkable amount of literature (see [12, 13] for a review of the field). In particular, many papers have addressed the case of random walks in dynamical environments. Apart from few papers in which special hypotheses are imposed on the form of the transition probabilities (implying that the process is reversible with respect to the stationary measure of the environment, see [10]) the authors have usually investigated the case in which the evolution of the environment is described by a Markov process with positive transition probabilities and the transition probabilities of the walk are close to constant (see, e.g., [4, 2] and references therein). While such a situation has recently been settled in great generality ([5]) the case of more complex local dynamics and/or far from constant transition probabilities is still wide open.

In this paper we address the first issue establishing conditions under which the CLT holds for a deterministic local evolution with strong chaotic properties. Such a CLT is established for almost all the initial conditions of the environment with respect to a (natural) stationary measure (this is commonly called a *quenched CLT*).

The deterministic dynamics is taken to be independent at each site (although our method can be easily extended to weakly interacting cases (cf. [8, 5])). The single site dynamics is a piecewise expanding one dimensional map. While multidimensional expanding dynamics could be treated similarly the case of Anosov map poses a real problem. Indeed the technique used to control the environment dynamics is borrowed from the study of coupled map lattices (more precisely from [9]) and the extension of such a technique to coupled Anosov systems is still missing. In general, the extension to more general dynamics (with substantially weaker ergodic properties) would be of interest, but to obtain results in this direction new ideas seem to be needed.

Note that the present strategy differs, in its probabilistic part, from the one used in [5]. In particular, it is not needed to prove the absolutely continuity of the invariant measure of the environment as seen from the particle with respect to the invariant measure of the environment in a fixed reference frame. We hope that this simplification may be helpful in treating more general cases.

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The plan of the paper is the following. In section 2 the system under investigation is explained in detail and the main result of the paper is precisely stated. This main result (Theorem 2.7) follows after estimating the asymptotic independence of two random walks in the same environment (Lemma 2.8). In Section 3 we study ergodic properties of the environment. Section 4 contains the proof of the annealed (averaged) invariance principle. In section 5 the proof of Lemma 2.8 is reduced to an estimate on the number of close encounters (Lemma 5.2). Lemma 5.2 is proven in Section 6.

Convention 1.1. *In this paper we will use C to designate a generic constant depending only on the quantities appearing in the Assumptions 1, 2, 3, 4 below. We will use instead $C_{a,b,c,\dots}$ for constants depending also from the parameters $a, b, c, \dots$. Consequently the actual numerical value of such constants may vary from one occurrence to the next. On the contrary we will use $C_1, C_2, \dots$, to designate constants whose value is fixed through the paper.*

2. MODEL AND RESULTS

Let $I = [0, 1]$ and $T : I \rightarrow I$ be a piecewise $\mathcal{C}^2$ topologically mixing map such that $|D_x T| \geq \lambda > 2$ for each $x \in I$ for which the derivative is well defined. Then $I^{\mathbb{Z}^d} =: \Theta$ is the space of environment (it is a measurable space with the product (Borel) σ -algebra $\mathcal{T}$) and $\theta \in \Theta$ is an environment on $\mathbb{Z}^d$. This environment evolves deterministically according to a map $F : \Theta \rightarrow \Theta$.

Assumption 1 (*Environment Dynamics*). *For each $\theta \in \Theta$*

$$(F(\theta))_q := T(\theta_q).$$

*That is, the evolution is independent at each site.*¹

The evolution of the environment can be thus seen as a deterministic Markov process on the space $\Theta^{\mathbb{N}} =: \Omega$ such that, for all $(\theta^n)_{n \in \mathbb{N}} := \theta \in \Omega$, $\theta^n := F^n(\theta^0)$. If μ_0 is the unique absolutely continuous invariant measure of T ,² then $\mu^e := \otimes_{p \in \mathbb{Z}^d} \mu_0$ is the natural invariant measure for F we are interested in. In fact, it is possible to show ([9]) that it is the only invariant measure in a reasonably large class of measures; see the precise statement below.

We consider a bounded increment random walk X_n in such an environment. More precisely, let $\Lambda := \{z \in \mathbb{Z}^d : \|z\| \leq C_0\}$ and $\Delta_n = X_{n+1} - X_n$, then the process is defined by the transition probabilities

$$(2.1) \quad \mathbb{P}(\{\Delta_n = z\} \mid X_n, \theta^0) = \pi_z(\tau^{X_n} \theta^n)$$

where $\pi_z \equiv 0$ for all $z \notin \Lambda$, $\pi_z(\theta)$ depends only on $\{\theta_q\}_{q \in \Lambda}$, and, for each $z \in \mathbb{Z}^d$, $(\tau^z \theta)_i := \theta_{i+z}$. We will be interested in the measure $\mathbf{P}_\nu$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ determined by the above process when the environment is started with the measure ν and the walk starts from zero. We will use the notation $\mathbf{P}^e$ for $\mathbf{P}_{\mu^e}$. Finally, we will use $\mathbb{E}$ for the expectation with respect to the latter measure and $\mathbb{E}_\nu$ for the expectations with respect to the process $\mathbf{P}_\nu$.

Assumption 2 (*Regularity*). *The functions $\{\pi_z\}_{z \in \Lambda}$ belong to $\mathcal{C}^1$.*

¹The case of weakly coupled maps can be treated similarly by using the techniques introduced in [9] and used here to study the present, simpler, case.

²For the existence and uniqueness (among measures absolutely continuous w.r.t. Lebesgue) see [1].

The next assumption depends on a parameter $\varepsilon > 0$.

Assumption 3 (*Perturbative regime*). *There exists $\{a_z\}_{z \in \Lambda} \subset \mathbb{R}_+$, $\sum_{z \in \Lambda} a_z = 1$, such that*

$$\|\pi_z - a_z\|_{C^1} \leq a_z \varepsilon.$$

In the following, when we will say “assumption 3 holds for ε_i ” we will mean that it holds with $\varepsilon = \varepsilon_i$. The values ε_i will be taken small enough for Theorem 2.2, Proposition 2.4, Lemma 2.5 and Lemma 3.5 to hold.

Assumption 4 (*Ellipticity*). *For each $l \in \mathbb{Z}^d \setminus \{0\}$, $\left| \sum_{z \in \Lambda} \pi_z e^{i\langle l, z \rangle} \right| \neq 1$.*

It is well known that to study the properties of X_n it is convenient to study the process of the environment as seen from the particle. In fact, such a process can be considered in several fashions of which the following will be relevant in the sequel.

2.1. Process of the environment as seen from the particle. Consider the process $\omega =: (\omega^n)_{n \in \mathbb{N}} \in \Omega$ described by the action of the Markov operator $S : L^\infty(\Theta) \rightarrow L^\infty(\Theta)$ defined by

$$(2.2) \quad Sf(\omega) := \sum_{z \in \Lambda} \pi_z(\omega) f \circ F(\tau^z \omega) =: \sum_{z \in \Lambda} S_z f.$$

Remark 2.1. *It is easy to verify that the process ω , $\omega^0 = \theta$, has the same distribution as the process $(\tau^{X_n} \theta^n)_{n \in \mathbb{N}}$, $\theta^0 = \theta$.*

We can then consider the measure $\mathbb{P}_\nu$ on Ω of the associated Markov process started with a measure ν .

In analogy with the techniques used in the study of coupled map lattices [9] it is then natural to restrict the space of measures on which S' acts.³ To this end we start by defining the following norms

$$(2.3) \quad \begin{aligned} |\mu| &:= \sup_{|\varphi|_{C^0(\Theta, \mathbb{R})} \leq 1} \mu(\varphi) \\ \|\mu\| &:= \sup_{i \in \mathbb{Z}^d} \sup_{|\varphi|_{C^0(\Theta, \mathbb{R})} \leq 1} \mu(\partial_{\theta_i} \varphi) \end{aligned}$$

We then consider the Banach space of complex valued measures⁴

$$(2.4) \quad \mathcal{B} := \{\mu \in \mathcal{M}(\Theta) : \|\mu\| < \infty\}.$$

It is easy to check that such measures have finite dimensional marginals absolutely continuous w.r.t. Lebesgue and the density are functions of bounded variations with variations bounded by the norm of the measure, moreover μ^e is the unique invariant measure for F belonging to $\mathcal{B}$, [9].

Theorem 2.2. *For each dynamics F satisfying assumption 1 and transition probabilities satisfying assumption 2, the operator S' , is a bounded operator on $\mathcal{B}$. In addition, there exists $\varepsilon_0 > 0$, depending on F , such that if assumption 3 holds for ε_0 , then there exists a unique invariant probability measure $\mu^w \in \mathcal{B}$ ($S' \mu^w = \mu^w$). This measure enjoys the following properties: There exists $\eta \in (0, 1)$ such that for each $\nu \in \mathcal{B}$ and local functions $\varphi, \phi \in C^0$ each depending only on L variables with the two sets of dependency having distance at least M*

³As usual the dual operator S' is defined as $S' \nu(f) = \nu(Sf)$ for all $f \in C^0$.

⁴By $\mathcal{M}(\Theta)$ we designate the set of complex valued finite Borel measures on Θ .

- (1) $|\nu(S^n \phi) - \mu^w(\phi)\nu(1)| \leq CL\eta^n |\phi|_\infty \|\nu\|$
 (2) $|\mu^w(\varphi\phi) - \mu^w(\varphi)\mu^w(\phi)| \leq CL\eta^{\frac{M}{2C_0}} |\varphi|_\infty |\phi|_\infty$.

The proof of the above theorem can be found in Section 3.2.

Remark 2.3. *Theorem 2.2 implies that the process $\mathbb{P}_{\mu^w}$ is a stationary (and ergodic) process.*

2.2. Annealed statistical properties.

Proposition 2.4. *For each dynamics F satisfying assumption 1 and transition probabilities satisfying assumption 2, if assumption 3 is satisfied for ε_0 (where $\varepsilon_0 > 0$ is as in Theorem 2.2), then there exists a vector $v \in \mathbb{R}^d$ and a matrix $\Sigma^2 \geq 0$ such that, for each probability measure $\nu \in \mathcal{B}$ we have*

$$\begin{aligned} \frac{1}{N} \mathbb{E}_\nu(X_N) &\rightarrow v \\ \frac{X_N - vN}{\sqrt{N}} &\Rightarrow \mathcal{N}(0, \Sigma^2) \quad \text{under } \mathbf{P}_\nu. \end{aligned}$$

Moreover there exists $C_1 > 0$ such that, setting $\tilde{X}_N := X_N - vN$, the following inequality holds, for all $N \in \mathbb{N}$ and $t \in \mathbb{R}^d$ such that $N \geq C_1$ and $\|t\| \leq N^{\frac{1}{6}}$,

$$\left| \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_N \rangle} \right) - e^{-\frac{1}{2} \langle t, \Sigma^2 t \rangle} \right| \leq C_1 (1 + \|t\|^3) N^{-\frac{1}{2}} \|\nu\|.$$

Finally, if Assumption 4 is also satisfied, then $\Sigma^2 > 0$.

Proof. Let us start noticing that

$$\frac{1}{N} \mathbb{E}_\nu(X_N) = \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}_\nu(\Delta_k) = \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}_\nu(\mathbb{E}_\nu(\Delta_k | \mathcal{F}_k)),$$

where $\mathcal{F}_k := \sigma\{\theta^0, X_1, \dots, X_k\}$. The relevance of the process as seen from the particle is due to the following fact:

$$(2.5) \quad \mathbb{E}_\nu(\Delta_k | \mathcal{F}_k) = \sum_{z \in \Lambda} z \pi_z(\tau^{X_k} \theta^k) = \sum_{z \in \Lambda} z \pi_z(\omega^k) =: g(\omega^k).$$

Thus

$$\frac{1}{N} \mathbb{E}_\nu(X_N) = \frac{1}{N} \sum_{k=0}^{N-1} \nu(g(\omega^k)) = \frac{1}{N} \sum_{k=0}^{N-1} [(S')^k \nu](g).$$

Accordingly, Theorem 2.2 implies

$$(2.6) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_\nu(X_N) = \mu^w(g) =: v.$$

To prove the CLT let $\tilde{\Delta}_n = \tilde{X}_{n+1} - \tilde{X}_n$, then

$$\mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_N \rangle} \right) = \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_{N-1} \rangle} \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \Delta_{N-1} - v \rangle} | \mathcal{F}_{N-1} \right) \right).$$

Since

$$\mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \Delta_k - v \rangle} | \mathcal{F}_k \right) = \sum_{z \in \Lambda} \pi_z(\tau^{X_k} \theta^k) e^{\frac{i}{\sqrt{N}} \langle t, z - v \rangle}$$

it is natural to introduce the operators, for all $t \in \mathbb{C}^d$,

$$(2.7) \quad \mathcal{M}_t h(\theta) := \sum_{z \in \Lambda} \pi_z(\theta) e^{\langle t, z - v \rangle} h(\tau^z F(\theta)) = \sum_{z \in \Lambda} e^{\langle t, z - v \rangle} S_z h.$$

Then,

$$\mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \Delta_k - v \rangle} \mid \mathcal{F}_k \right) = (\mathcal{M}_{it/\sqrt{N}} 1)(\tau^{X_k} \theta^k),$$

and the reader can then check, by induction, the formula

$$(2.8) \quad \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_N \rangle} \right) = \nu(\mathcal{M}_{it/\sqrt{N}}^N 1).$$

The operator $\mathcal{M}_t'$ acting on the space $\mathcal{B}$ is an analytic perturbation of the operator $S' = \mathcal{M}_0'$. Unfortunately, S' does not have a nice spectrum on $\mathcal{B}$, so in order to apply usual perturbation theory, it is necessary to lift the dynamics to an appropriate space in the spirit of [3]. We do so in section 3.3 where we prove the following result.

Lemma 2.5. *Under the assumptions of Theorem 2.2 there exists $C_2 > 0$ and a function α_t analytic near $\{\|t\| \leq C_2\}$ such that for each $n \in \mathbb{N}$, probability measure $\nu \in \mathcal{B}$ and local function f depending on L variables we have*

$$\begin{aligned} |\nu(\mathcal{M}_t^n f)| &\leq CL |\alpha_t^n| \|f\|_\infty \|\nu\| \\ \nu(\mathcal{M}_t^n 1) &= \alpha_t^n (1 + \mathcal{O}(t\|\nu\|)) + \mathcal{O}(\eta^n \|\nu\|). \end{aligned}$$

Moreover $\alpha_0 = 1$, $\dot{\alpha}_0 = 0$ and $\ddot{\alpha}_0 > 0$ (the “dot” stands for the derivatives with respect to t).

Using Lemma 2.5 and setting $\Sigma^2 := \ddot{\alpha}_0$, we have

$$\nu(\mathcal{M}_{it/\sqrt{N}}^N 1) = \alpha_{it/\sqrt{N}}^N (1 + \mathcal{O}(tN^{-\frac{1}{2}} \|\nu\|)) + \mathcal{O}(\eta^N \|\nu\|).$$

We can finally compute, for $\|t\| \leq CN^{\frac{1}{6}}$ and N large enough,⁵

$$\begin{aligned} \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_N \rangle} \right) &= \alpha_{it/\sqrt{N}}^N + \mathcal{O} \left(\frac{\|t\|}{\sqrt{N}} \|\nu\| \right) \\ &= \left(1 - \frac{1}{2N} \langle t, \Sigma^2 t \rangle + \mathcal{O}(\|t\|^3 N^{-\frac{3}{2}}) \right)^N + \mathcal{O} \left(\frac{1 + \|t\|}{\sqrt{N}} \|\nu\| \right) \\ &= e^{-\frac{1}{2} \langle t, \Sigma^2 t \rangle + \mathcal{O}(\|t\|^3 N^{-\frac{1}{2}})} + \mathcal{O} \left(\frac{1 + \|t\|}{\sqrt{N}} \|\nu\| \right), \end{aligned}$$

from which Proposition 2.4 follows. $\square$

Next, we need a large deviations estimate.

Lemma 2.6. *Under the assumptions of Theorem 2.2 there exists $a_0 > 0$ such that for each $\nu \in \mathcal{B}$, $n, m \in \mathbb{N}$ and $a \in (0, a_0)$ the following holds true*

$$\mathbf{P}_\nu \left(\left\{ \left| \frac{1}{m} (\tilde{X}_{n+m} - \tilde{X}_n) \right| \geq a \right\} \right) \leq C e^{-C a^2 m} (\|\nu\| + 1).$$

Proof. Again this large deviation result can be obtained by perturbation theory of the operator S' . Indeed, for each $w \in \mathbb{R}^d$, $\|w\| = 1$ and $t \in \mathbb{R}$,

$$\begin{aligned} \mathbf{P}_\nu \left(\left\{ \frac{1}{m} \langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle \geq a \right\} \right) &\leq \mathbb{E}_\nu \left(e^{t(\langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle - am)} \right) \\ &= e^{-tam} [(S')^n \nu](\mathcal{M}_{tw}^m 1). \end{aligned}$$

⁵If $\|t\| \geq CN^{\frac{1}{6}}$ the last statement of Proposition 2.4 is obvious: the left hand side is ≤ 2 .

Since by Theorem 2.2 $\sup_{n \in \mathbb{N}} \|(S')^n \nu\| \leq C\|\nu\|$, we can apply Lemma 2.5 and obtain, for $t \leq C_2$,

$$\begin{aligned} \mathbf{P}_\nu \left(\left\{ \frac{1}{m} \langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle \geq a \right\} \right) &\leq C e^{-tam} \alpha_{tw}^m (1 + C|t|\|\nu\|) + C\eta^m \|\nu\| \\ &\leq C e^{-tam} \left(1 + \frac{t^2}{2} \langle w, \Sigma^2 w \rangle + \mathcal{O}(|t|^3) \right)^m (1 + \|\nu\|) + C\eta^m \|\nu\| \\ &\leq C e^{-tam + \frac{mt^2}{2} \langle w, \Sigma^2 w \rangle + \mathcal{O}(m|t|^3)} (1 + \|\nu\|) + C\eta^m \|\nu\|. \end{aligned}$$

Finally, choosing $t = \frac{a}{\langle w, \Sigma^2 w \rangle}$ and a_0 so small that the term $\mathcal{O}(t^3)$ is small with respect to $\frac{a_0^2}{2\langle w, \Sigma^2 w \rangle}$, $\frac{a_0}{\langle w, \Sigma^2 w \rangle} < C_2$ and $e^{-\frac{a_0^2}{2\langle w, \Sigma^2 w \rangle}} \geq \eta$

$$\mathbf{P}_\nu \left(\left\{ \frac{1}{m} \langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle \geq a \right\} \right) \leq C e^{-Ca^2 m} (1 + \|\nu\|).$$

We then conclude by noticing that the above estimate for all the w in the set $\{\pm e_i\}_{i=1}^d$, where $\{e_i\}_{i=1}^d$ is the standard base of $\mathbb{R}^d$, implies the Lemma. $\square$

2.3. Main result: Quenched C.L.T. Let $\mathbf{P}_\theta$ be the measure $\mathbf{P}^e$ conditioned to starting the environment in the configuration θ . We will use $\mathbb{E}_\theta$ for the expectation with respect to $\mathbf{P}_\theta$.

Theorem 2.7. *For each dynamics F satisfying assumption 1 and transition probabilities satisfying assumptions 2 and 4, if assumption 3 is satisfied for ε_1 (where $\varepsilon_0 \geq \varepsilon_1 > 0$ is as in Lemma 3.5), then (using the same notations as in Proposition 2.4), for μ^e almost all $\theta \in \Theta$ the following holds*

(a) $\frac{1}{N} X_N \rightarrow v$ $\mathbf{P}_\theta$ a.s.;

(b) $\frac{X_N - vN}{\sqrt{N}} \Rightarrow \mathcal{N}(0, \Sigma^2)$ under $\mathbf{P}_\theta$.

Proof. Lemma 2.6 implies the bound $\mathbf{P}^e(\{|N^{-1}X_N - v| \geq \varepsilon\}) \leq C e^{-C\varepsilon^2 N}$; (a) follows then by applying Borel-Cantelli.

To prove (b) let $\alpha \in (0, 1)$ be a number to be specified later. Combining Lemma 2.6 and Borel-Cantelli Lemma we see that for any $\delta > 0$, $\mathbf{P}^e$ -almost surely for any k and $0 \leq j \leq 2^{(1-\alpha)k}$ we have

$$(2.9) \quad \max_{m \in [2^k + j2^{\alpha k}, 2^k + (j+1)2^{\alpha k}]} \left| \tilde{X}_m - \tilde{X}_{2^k + j2^{\alpha k}} \right| \leq C_{\theta, X} 2^{(\frac{\alpha}{2} + \delta)k}$$

By Fubini Theorem for almost every θ (2.9) holds $\mathbf{P}_\theta$ almost surely. Therefore it is enough to prove the convergence along the subsequence $n_{jk} = 2^k + j2^{\alpha k}$ where $0 \leq j \leq 2^{(1-\alpha)k}$.

To conclude it suffices to prove that there exists $\beta > 0$ and $b \in \mathbb{N}$ such that for each smooth function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ compactly supported in a box of size L the following inequality holds

$$(2.10) \quad \mathbb{E} \left(\left| \mathbb{E}_\theta(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) \right|^2 \right) \leq C_L |\varphi|_{C^b} N^{-\beta},$$

where $\mathbb{E}_{\mathcal{N}(0, \Sigma^2)}$ is the expectation with respect to the Gaussian measure $\mathcal{N}(0, \Sigma^2)$. Indeed, denote

$$\xi_{jk} = \mathbb{E}_\theta \left(\varphi \left(\frac{\tilde{X}_{n_{jk}}}{\sqrt{n_{jk}}} \right) \right).$$

Then (2.10) and Chebyshev inequality imply

$$(2.11) \quad \mathbb{P}(\{|\xi_{jk} - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)| \geq \varepsilon\}) \leq C_L |\varphi|_{\mathcal{C}^b} \varepsilon^{-2} n_{jk}^{-\beta}.$$

Hence, by finally choosing α such that $\alpha + \beta > 1$, $\xi_{jk} \rightarrow \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)$ almost surely. Next, choose a family φ_m which is dense in $\mathcal{C}_0^0(\mathbb{R}^d)$. Then, for almost every θ , we have

$$\mathbb{E}_\theta \left(\varphi_m \left(\frac{\tilde{X}_{n_{jk}}}{\sqrt{n_{jk}}} \right) \right) \rightarrow \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi_m)$$

for all m . Then, for any such θ

$$\mathbb{E}_\theta \left(\varphi \left(\frac{\tilde{X}_{n_{jk}}}{\sqrt{n_{jk}}} \right) \right) \rightarrow \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)$$

for any continuous compactly supported φ proving (b).

The result is then proved provided (2.10) is true. It turns out that (2.10) can be conveniently interpreted in terms of two independent walks X_N, Y_N in the same environment. In fact, calling $\mathbb{E}^2$ the expectation with respect to such a process it follows

$$\begin{aligned} \mathbb{E} \left(\left| \mathbb{E}_\theta(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) \right|^2 \right) &= \mathbb{E}^2(\varphi(N^{-\frac{1}{2}} \tilde{X}_N) \varphi(N^{-\frac{1}{2}} \tilde{Y}_N)) \\ &\quad - 2\mathbb{E}(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) + \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)^2 \\ &= \mathbb{E}^2(\varphi(N^{-\frac{1}{2}} \tilde{X}_N) \varphi(N^{-\frac{1}{2}} \tilde{Y}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)^2 + \mathcal{O}(L^d \|\varphi\|_{\mathcal{C}^{d+1}} N^{-\beta}). \end{aligned}$$

where we have used the quantitative estimate in the Proposition 2.4.⁶

We have thus reduced the proof of the theorem to proving the following.

Lemma 2.8.

$$\left| \mathbb{E}^2(\varphi(N^{-\frac{1}{2}} \tilde{X}_N) \varphi(N^{-\frac{1}{2}} \tilde{Y}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)^2 \right| \leq C \|\varphi\|_{\mathcal{C}^b} N^{-\beta}.$$

Lemma 2.8 is proved in Section 5. □

3. PROOFS: ENVIRONMENT

In this section we establish all the needed properties of the environment dynamics. The basic idea is to prove that the environment enjoys very strong mixing properties. Our first aim is to prove Theorem 2.2, that is exponential decay of space-time correlations.

Note that $S1 = 1$, hence

$$(3.1) \quad |S\mu| \leq |\mu|,$$

⁶If $\hat{\varphi}$ is the Fourier transform of φ , then Proposition 2.4 yields, for each $\rho < \frac{1}{6}$ and $b \in \mathbb{N}$,

$$\begin{aligned} \left| \mathbb{E}(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) \right| &\leq C \int_{\|t\| \geq N^\rho} |\hat{\varphi}(t)| dt + C \int_{\|t\| \leq N^\rho} \|t\|^3 N^{-\frac{1}{2}} |\hat{\varphi}(t)| dt \\ &\leq CL^d \|\varphi\|_{\mathcal{C}^b} \int_{|x| \geq N^\rho} |x|^{-b+d-1} dx + CL^d \|\varphi\|_{\mathcal{C}^0} N^{-\frac{1}{2}} \int_{|x| \leq N^\rho} |x|^{3+d-1} dx, \end{aligned}$$

which gives the advertised result provided $b > d$ and ρ is chosen small enough.

that is S is a contraction in the $|\cdot|$ norm. In addition, it is possible to prove (although we will not use it here) that there exists $A, B > 0$ and $\sigma \in (2\lambda^{-1}, 1)$ such that, for each $n \in \mathbb{N}$ and $\mu \in \mathcal{B}$,⁷

$$\|(S')^n \mu\| \leq A\sigma^n \|\mu\| + B|\mu|.$$

For finitely many sites the above estimate would suffice to prove that the operator S' is quasi-compact and this, together with the topologically mixing assumption, would imply the existence of a spectral gap. Unfortunately, such a proof is based on the compactness of the unit ball $\{\mu \in \mathcal{B} : \|\mu\| \leq 1\}$ in the topology of the $|\cdot|$ norm which fails when one considers infinitely many sites.

The obvious idea is to use explicitly the fact that the dynamics in different sites are independent, hence the system has a product structure, yet this is a subtle issue. To understand better the situation, let us recall few facts about the single site systems. At each site we have the dynamical system $(I = [0, 1], T)$. Let us consider the norm in $\mathcal{M}(I)$ given by

$$\|\nu\|_0 = \sup_{|\varphi|_{C^0} \leq 1} \mu(\varphi'),$$

where φ' is the derivative of φ . The Banach space $B = \{\nu \in \mathcal{M} : \|\nu\|_0 < \infty\}$ consists of measures absolutely continuous with respect to the Lebesgue measure $m_{\mathcal{L}}$. In addition, if $d\nu = h dm_{\mathcal{L}}$, then the density h is a function of bounded variation and $\|h\|_{BV} = \|\nu\|$.⁸ By a change of variable one can compute that, if $d\nu = h dm_{\mathcal{L}}$, then $d(T'\nu) = (\mathcal{L}h)dm_{\mathcal{L}}$, where the operator $\mathcal{L}$ is defined as

$$\mathcal{L}h(x) = \sum_{y \in T^{-1}(x)} |D_y T|^{-1} h(y).$$

The operator $\mathcal{L}$ is often called the Ruelle-Perron-Frobenius transfer operator. It is well known that, if T is topologically mixing, then the operator $\mathcal{L}$, acting on BV has 1 as a simple eigenvalue (corresponding to the unique invariant measure absolutely continuous with respect to Lebesgue) and enjoys a spectral gap, that is there exists $\eta_0 \in (0, 1)$ such that the rest of the spectrum is strictly contained in a disk of radius η_0 (see, e.g., [1] for details). Clearly the above implies that T' has a spectral gap when acting on B . Unfortunately, it turns out that the tensor products of B is a too small a space to be really useful for our purposes.⁹ This is the reason why we have introduced the spaces $\mathcal{B}$ which is a generalization of measures with density of bounded variations to the infinite dimensional setting. Yet on such a space S' does not behave very well and we will use an abstract covering space on which the dynamics will exhibit a spectral gap.

More precisely, we would like to introduce a Banach space $\overline{\mathcal{B}}$ and two (possibly only partially defined) maps $\Psi : \mathcal{B} \rightarrow \overline{\mathcal{B}}$ and $\text{Pr} : \overline{\mathcal{B}} \rightarrow \mathcal{B}$ and an operator $\mathbf{S} : \overline{\mathcal{B}} \rightarrow \overline{\mathcal{B}}$ such that the dynamics of the latter covers the dynamics of S' as illustrated by the

⁷See (2.4) for the definition of $\mathcal{B}$.

⁸The equality of the norms follows from the usual weak definition of BV , the fact that the measures must be absolutely continuous can be easily proved by approximating a measure with finite norm by one with a smooth density (just use a mollifier) and remembering that the unit ball of BV is compact in L^1 . See [8] for more details.

⁹The problem is already present for two sites since $BV(I) \otimes BV(I) \neq BV(I^2)$.

following commutative diagram

$$(3.2) \quad \begin{array}{ccc} \bar{\mathcal{B}} & \xrightarrow{\mathbf{S}^n} & \bar{\mathcal{B}} \\ \Psi \uparrow & & \downarrow \text{Pr} \\ \mathcal{B} & \xrightarrow{(S')^n} & \mathcal{B} \end{array}$$

We will first define the space $\bar{\mathcal{B}}$ and the map $\mathbf{S}$. Then we will prove Theorem 2.2 by proving that $\mathbf{S}$ has a spectral gap on $\bar{\mathcal{B}}$. Next we will obtain others, more refined, results by using the same strategy (albeit applied to different operators).

3.1. Covering dynamics. First we define the above mentioned abstract space. Let $\bar{\mathcal{B}} := \mathbb{C} \times [\times_{p \in \mathbb{Z}^d} \mathcal{B}_p]$ where

$$\mathcal{B}_p := \{\mu \in \mathcal{B} : \mu(\varphi) = 0 \ \forall \varphi \in \mathcal{C}^0(\Theta) \text{ that do not depend on } \theta_p\}.$$

The vector space $\bar{\mathcal{B}}$ is a Banach space when equipped with the norm

$$\|\mu\| := \sup\{|c_\mu|, \|\mu_p\| : p \in \mathbb{Z}^d\}.$$

Here we use the notational convention that an element $\mu \in \bar{\mathcal{B}}$ has components $c_\mu \in \mathbb{C}$ and $\bar{\mu} := (\mu_p)_p$ with $\mu_p \in \mathcal{B}_p$.¹⁰

Next we define a projection $\text{Pr} : \bar{\mathcal{B}} \rightarrow \mathcal{B}$ and a map $\Psi : \mathcal{B} \rightarrow \bar{\mathcal{B}}$ allowing to transfer objects between the two spaces.

Let $\mu_* \in \mathcal{B}$ be a fixed probability measure on I then $m = \otimes_{\mathbb{Z}^d} \mu_*$ is a product probability measure on Θ . For each $\mu = (c_\mu, (\mu_p)_p) \in \bar{\mathcal{B}}$ and local function f we define

$$(3.3) \quad \text{Pr } \mu(f) := c_\mu m(f) + \sum_{p \in \mathbb{Z}^d} \mu_p(f),$$

which makes clear in which sense $\bar{\mathcal{B}}$ “covers” $\mathcal{B}$ (or $\mathcal{M}(\Theta)$).

Remark 3.1. Note that, although $\text{Pr } \mu(f)$ is well defined on each local function, $\text{Pr } \mu$ is not necessarily a measure. Let $\bar{\mathcal{B}}_M \subset \bar{\mathcal{B}}$ be such that the elements of $\text{Pr } \bar{\mathcal{B}}_M$ give rise to bounded linear functionals on the space of local functions, and hence identify uniquely a measure.¹¹ We will call such a measure $\text{Pr } \mu$.

The choice of the map Ψ is quite arbitrary, we will fix a convenient one. Consider a strict total ordering $\prec$ of $\mathbb{Z}^d$ such that $0 \prec p$ for each $p \in \mathbb{Z}^d \setminus \{0\}$ and the sets $\{q : q \prec p\}$ are finite for each $p \in \mathbb{Z}^d$.¹²

Let q_+ be the successor of q (that is, $q \prec q_+$ and there are no $q' \in \mathbb{Z}^d$ such that $q \prec q' \prec q_+$). For each $q \in \mathbb{Z}^d$ we can then consider the σ -algebra $\mathcal{F}_q^0$ determined by all the variables $\theta_{q'}$ with $q \preceq q'$, hence $\mathcal{F}_0^0$ is the complete σ -algebra. Next, for each $f \in \mathcal{C}^0(\Theta)$ and $q \in \mathbb{Z}^d$, define the operator $J_q f = m(f | \mathcal{F}_q^0) - m(f | \mathcal{F}_{q_+}^0)$.

¹⁰For example, if ν_p, ν'_p ($p \in \mathbb{Z}^d$) are probability measures on Θ such that $\nu_p = \nu'_p$ for all $p \neq q$, and we set $\nu := \otimes_{p \in \mathbb{Z}^d} \nu_p$, $\nu' := \otimes_{p \in \mathbb{Z}^d} \nu'_p$, then $\nu - \nu' \in \bar{\mathcal{B}}_q$.

¹¹Since the local functions are dense in the continuous ones by the Stone-Weierstrass theorem.

¹²For example, one can start from zero and spiral out on larger and larger cubical shells.

For each local function f we can write¹³

$$f = m(f) + \sum_{q \in \mathbb{Z}^d} J_q(f)$$

Accordingly, for each $\mu \in \mathcal{B}$ we define $\mu_q(f) := J'_q \mu(f) \in \mathcal{B}_q$, and the lift

$$\Psi(\mu) := (\mu(1), (J'_q \mu)_q) .$$

Note that Ψ is a bounded operator. Indeed if $q \prec p$, then

$$|J'_q \mu(\partial_{\theta_p} \varphi)| \leq |\mu| |J_q(\partial_{\theta_p} \varphi)|_\infty \leq 2|\mu| \|\mu_*\| |\varphi|_\infty .$$

If $q \succ p$, then

$$|J'_q \mu(\partial_{\theta_p} \varphi)| \leq |\mu(\partial_{\theta_p} J_q \varphi)| \leq \|\mu\| |J_q(\varphi)|_\infty \leq 2\|\mu\| |\varphi|_\infty .$$

Finally, for $q = p$, we have $|J'_q \mu(\partial_{\theta_p} \varphi)| \leq (|\mu| \|\mu_*\| + \|\mu\|) |\varphi|_\infty$. In other words there exists $C_3 > 0$, depending on the choice of μ_* , such that

$$(3.4) \quad \|\Psi(\mu)\| \leq C_3 \|\mu\| .$$

Clearly, $\Psi(\mathcal{B}) \subset D$ and for each $\mu \in \mathcal{B}$ it holds true

$$\Pr(\Psi(\mu)) = \mu .$$

Now that we know how to lift measures, we can address the dynamics.

For all $z \in \mathbb{Z}^d$, $\tau^z \mu_q(\varphi) := \mu_q(\varphi \circ \tau^z) = 0$ if φ does not depend on θ_{q-z} . Thus we can define the decomposition for $\tau^z \mu$ via the decomposition $\mu_q = J'_q \mu$ of μ :

$$\tau^z \mu = \sum_{q \in \mathbb{Z}^d} (\tau^z \mu)_q := \sum_{q \in \mathbb{Z}^d} \tau^z \mu_{q-z} .$$

Setting $\Lambda_1 = \cup_{z \in \Lambda} \tau^{-z} \Lambda$, we can define the covering dynamics $\mathbf{S}\mu$ by $(c_\mu, \bar{S}\bar{\mu} + \bar{\zeta}c_\mu)$ where $\bar{\zeta} := (\zeta_q)$ with $\zeta_q := J'_q S' m =: J'_q \zeta$ and

$$(3.5) \quad (\bar{S}\bar{\mu})_p = \begin{cases} \sum_{z \in \Lambda} S'_z \mu_{p-z} + \sum_{q \in \Lambda_1} \sum_{z \in \Lambda} J'_p \hat{S}'_z \mu_{q-z} & \text{for each } p \notin \Lambda_1, \\ \sum_{z \in \Lambda} A'_z \mu_{p-z} + \sum_{q \in \Lambda_1} \sum_{z \in \Lambda} J'_p \hat{S}'_z \mu_{q-z} & \text{for each } p \in \Lambda_1, \end{cases}$$

where $S_z f(\omega) := \pi_z(\omega) f \circ F \circ \tau^z(\omega)$, $A_z f(\omega) := a_z f \circ F \circ \tau^z(\omega)$ and $\hat{S}_z := S_z - A_z$. It is easy to check the following.

Lemma 3.2. *The operator $\mathbf{S}$ is well defined as a bounded operator from $\bar{\mathcal{B}}$ to $\bar{\mathcal{B}}$. For each $\mu \in \bar{\mathcal{B}}$ and continuous local function f we have $\Pr(\mathbf{S}\mu)(f) = \Pr(\mu)(Sf)$ which implies that for each $n \in \mathbb{N}$ and $\mu \in \mathcal{B}$ we have $\Pr \mathbf{S}^n \Psi \mu = S^{n'} \mu$*

We have thus established a setting in which the commutative diagram (3.2) holds true.

¹³As f is local there exists a box $\Lambda_* \subset \Theta$ such that f depends only from the variables $\{\theta_q : q \in \Lambda_*\}$, but this means that the sum consists only of finitely many terms.

3.2. Mixing properties of the environment. We have now the necessary machinery to deal with the statistical properties of the environment.

Proof of Theorem 2.2 . Let us first discuss the environment dynamics F . Its basic properties are described by the so called Lasota-Yorke inequalities asserting that there exists $B > 0$ such that, for all $n \in \mathbb{N}$,¹⁴

$$(3.6) \quad \begin{aligned} |F'\mu| &\leq |\mu| \\ \|(F')^n \mu\| &\leq (2\lambda^{-1})^n \|\mu\| + B|\mu|. \end{aligned}$$

Note that the above implies that $\{\|(F')^n \mu\|\}_{n \in \mathbb{N}}$ is bounded.

Lemma 3.3. *There exists $\eta_* \in (\eta_0, 1)$ such that, for each $p \in \mathbb{Z}^d$, $\mu \in \mathcal{B}_p$,*

$$\|(F')^n \mu_q\| \leq C\eta_*^n \|\mu\|.$$

Proof. For each local function $\varphi \in \mathcal{C}^0$, we can define $\varphi_{\theta_{\neq q}}^n(\xi) := \varphi(\theta^{n,\xi})$, where $\theta_p^{n,\xi} = T^n \theta_p$ for each $p \neq q$ while $\theta_q^{n,\xi} = \xi$,

$$\begin{aligned} |(F')^n \mu_q(\varphi)| &= \left| \mu_q \left(\partial_{\theta_q} \int_0^1 [\chi_{[0, \theta_q]}(\xi) - \theta_q] \varphi_{\theta_{\neq q}}^n(T^n \xi) d\xi \right) \right| \\ &\leq \|\bar{\mu}\| \cdot \left| \int_0^1 \mathcal{L}^n [\chi_{[0, \theta_q]}(\xi) - \theta_q] \cdot \varphi_{\theta_{\neq q}}^n(\xi) d\xi \right|_{\infty} \\ &\leq C\|\bar{\mu}\| \|\mathcal{L}^n [\chi_{[0, \theta_q]} - \theta_q]\|_{BV} |\varphi|_{\infty} \leq C\|\bar{\mu}\| \eta_0^n |\varphi|_{\infty}, \end{aligned}$$

by the spectral gap of $\mathcal{L}$ and the fact that $m_{\mathcal{L}}(\chi_{[0, \theta_q]} - \theta_q) = 0$, i.e. it is a zero average function. Then, by the Lasota-Yorke inequality,

$$\begin{aligned} \|(F')^{j+k} \mu_q\| &\leq (2\lambda^{-1})^j \|(F')^k \mu_q\| + B|(F')^k \mu_q| \\ &\leq (2\lambda^{-1})^j [(2\lambda^{-1})^k \|\mu_q\| + B|\mu|] + BC\eta_0^k \|\mu_q\| \\ &\leq [(2\lambda^{-1})^{j+k} + (2\lambda^{-1})^j B + BC\eta_0^k] \|\mu_q\|. \end{aligned}$$

The result follows by optimizing the choice of $j + k = n$. $\square$

Lemma 3.4. *Multiplication by a $\mathcal{C}^1$ local function is a bounded operator on $\mathcal{B}$.*

Proof. For any smooth local functions ψ, ϕ , $|\psi|_{\infty} \leq 1$ we have

$$(3.7) \quad |\nu(\phi \cdot \partial_{\theta_i} \psi)| = \left| \nu \left(\partial_{\theta_i} \int_0^{\theta_i} (\phi \partial_{\theta_i} \psi) \right) \right| \leq \|\nu\| \left| \int_0^{\theta_i} \phi \partial_{\theta_i} \psi \right|_{\infty} \leq 3\|\nu\| |\phi|_{\mathcal{C}^1}.$$

$\square$

¹⁴The first is trivial since $|F'\mu(\varphi)| = |\mu(\varphi \circ F)| \leq |\mu| |\varphi|_{\infty}$. For the second, given any smooth local function φ , let $\varphi_{\theta_{\neq q}}(\xi) := \varphi(\theta^{\xi})$, where $\theta_p^{\xi} = T\theta_p$ for each $p \neq q$ while $\theta_q^{\xi} = \xi$. Next introduce a function ϕ , piecewise linear in the variable θ_q such that $\varphi_{\theta_{\neq q}}(\theta_p) - \phi(\theta) = 0$ for each θ_p on the discontinuity values of T . By construction $\varphi_{\theta_{\neq q}}(T\theta_p) - \phi(F\theta)$ is then a Lipschitz function in θ_p , thus

$$\begin{aligned} |F'\mu(\partial_{\theta_q} \varphi)| &= |\mu((\partial_{\theta_q}(\varphi - \phi)) \circ F)| + |\mu| |\partial_{\theta_q} \phi|_{\infty} \leq |\mu| (\partial_{\theta_q} |D_{\theta_q} T|^{-1}(\varphi - \phi) \circ F)| + C|\mu| |\varphi|_{\infty} \\ &\leq \|\mu\| |D_{\theta_q} T|^{-1}(\varphi - \phi) \circ F|_{\infty} + C|\mu| |\varphi|_{\infty} \leq [2\lambda^{-1} \|\mu\| + C|\mu|] |\varphi|_{\infty}. \end{aligned}$$

The above yields $\|F'\mu\| \leq 2\lambda^{-1} \|\mu\| + C|\mu|$ which iterated yields the wanted result with $B = (1 - 2\lambda^{-1})^{-1}C$. See [1, 8] if more details are needed.

To use the above facts, it is convenient to introduce a more compact notation for the pieces that make up the operator $\bar{S}$. Let $\mathbb{1}_A : \mathbb{Z}^d \rightarrow \{0, 1\}$ be the characteristic function of the set $A \subset \mathbb{Z}^d$. Then define the operators $K_{z,p,q,\sigma} : \mathcal{B} \rightarrow \mathcal{B}$ by $K_{z,p,q,0} := \mathbb{1}_{\{p\}}(q+z)A'_z$ and $K_{z,p,q,1} := \mathbb{1}_{\Lambda_1^c}(p)\mathbb{1}_{\{p\}}(q+z)\hat{S}'_z + \mathbb{1}_{\Lambda_1}(q+z)J'_p\hat{S}'_z$. With this notation (3.5) can be rewritten as

$$(\bar{S}\bar{\mu})_p = \sum_{z \in \Lambda} \sum_{\sigma \in \{0,1\}} \sum_{q \in \mathbb{Z}^d} K_{z,p,q,\sigma} \mu_q.$$

Hence, iterating,

$$(3.8) \quad (\bar{S}^n \bar{\mu})_{q_0} = \sum_{z_1, \dots, z_n \in \Lambda} \sum_{\sigma_1, \dots, \sigma_n \in \{0,1\}} \sum_{q_1, \dots, q_n \in \mathbb{Z}^d} K_{z_1, q_0, q_1, \sigma_1} \cdots K_{z_n, q_{n-1}, q_n, \sigma_n} \mu_{q_n}.$$

By Assumption 3, Lemma 3.3, Lemma 3.4 and the inequalities (3.6) follows that there exists a constant $C_4 > 0$, depending only on F and π_z , such that $\sum_q \|K_{z,p,q,1} \mu_q\| \leq C_4 \varepsilon a_z \|\bar{\mu}\|$ and

$$\sum_{q_1, \dots, q_n \in \mathbb{Z}^d} \|K_{z_1, q_0, q_1, 0} \cdots K_{z_n, q_{n-1}, q_n, 0} \mu_{q_n}\| \leq C_4 a_{z_1} \cdots a_{z_n} \eta_*^n \|\bar{\mu}\|$$

Accordingly, if $C_4^2 \varepsilon + \eta_* < 1$, then there exists $n_* \in \mathbb{N}$ and $\eta \in (\eta_*, 1)$ such that $C_4(\eta_* + C_4^2 \varepsilon)^{n_*} \leq \eta^{n_*} < 1$. This means that every $z_1, \dots, z_{n_*}$ term in (3.8) will be smaller than $\eta^{n_*} a_{z_1} \cdots a_{z_{n_*}} \|\bar{\mu}\|$, hence for all $n \in \mathbb{N}$,

$$(3.9) \quad \|(\bar{S}^n \bar{\mu})\| \leq C_4 \eta^{n-n_*} \|\bar{\mu}\|.$$

Since $\mathbf{S}^n \boldsymbol{\mu} = (c_\mu, \bar{S}^n \bar{\mu} + c_\mu \sum_{k=1}^{n-1} \bar{S}^k \bar{\zeta})$ and the series $\bar{\zeta}_* = \sum_{k=1}^{\infty} \bar{S}^k \bar{\zeta}$ converges by (3.9), it follows that $\boldsymbol{\mu}^w := (1, \bar{\zeta}_*)$ is an invariant vector for $\mathbf{S}$. In addition

$$(3.10) \quad \|\mathbf{S}^n \boldsymbol{\mu} - c_\mu \boldsymbol{\mu}^w\| \leq C \eta^n \|\boldsymbol{\mu}\|,$$

That is the operator $\mathbf{S}$ on $\bar{\mathcal{B}}$ has one as a simple maximal eigenvalue and a spectral gap. From this result we can obtain the decay of temporal correlation simply by projecting down to $\mathcal{B}$. Indeed, let μ be a probability measure and ϕ be a smooth local function depending only on the sites $A \subset \mathbb{Z}^d$ and let L be the cardinality of A , then, by Lemma 3.2 and (3.10), (3.4),

$$(3.11) \quad \begin{aligned} |\mu(\phi \circ S^n) - \Pr(\boldsymbol{\mu}^w)(\phi)| &= |\Pr(\mathbf{S}^n(\Psi(\mu) - \boldsymbol{\mu}^w))(\phi)| \\ &= \left| \sum_{q \in A} (\mathbf{S}^n(\Psi(\mu) - \boldsymbol{\mu}^w))_q(\phi) \right| \\ &\leq \sum_{q \in A} \|(\mathbf{S}^n(\Psi(\mu) - \boldsymbol{\mu}^w))_q\| |\phi|_\infty \leq CL \eta^n (\|\mu\| + C) |\phi|_\infty. \end{aligned}$$

Thus, remembering (3.1), $\boldsymbol{\mu}^w \in \bar{\mathcal{B}}_M$, that is it gives rise to a bounded linear functional on local functions. Accordingly, we can define the measure $\mu^w = \Pr(\boldsymbol{\mu}^w)$ which will be invariant by S' . Equation (3.11) gives then the temporal correlation decay for such a measure.

To have the spatial decay of correlations note that if φ and ϕ are supported at a distance M , then their support, under the dynamics, grows at most linearly in time, thus it will take a time $\frac{M}{2C_0}$ before the supports have a common variable.

Accordingly, since $\varphi\phi$ depends on $2L$ variables, (3.11) (applied repeatedly to the product measure m) implies

$$\begin{aligned}\mu^w(\varphi\phi) &= (S')^{M/2C_0}m(\varphi\phi) + \mathcal{O}(L\eta^{M/2C_0}|\varphi\phi|_\infty) \\ &= m(S^{M/2C_0}\varphi)m(S^{M/2C_0}\phi) + \mathcal{O}(L\eta^{M/2C_0}|\varphi\phi|_\infty) \\ &= \mu^w(g)\mu^w(\phi) + \mathcal{O}(L\eta^{M/2C_0}|\varphi\phi|_\infty).\end{aligned}$$

□

3.3. Perturbation Theory. In this section we prove Lemma 2.5.

We deal with operators of the type $\mathcal{M}_t f := \sum_{z \in \Lambda} S_z(e^{\langle t, z-v \rangle} f)$ where $t \in \mathbb{C}^d$. The problem is to study the spectrum for small t .¹⁵

Proof of Lemma 2.5. First of all we need to lift the operator to our covering space. The obvious solution is to define $\mathbf{M}_t \boldsymbol{\mu}$ by

$$\left(c_\mu m(\mathcal{M}_t 1) + \sum_{p \in \Lambda_1} \sum_{z \in \Lambda} e^{\langle t, z-v \rangle} \mu_{p-z}(\hat{S}_z 1), \bar{S}_t \bar{\mu} + \bar{\zeta} c_\mu \right)$$

where $\bar{\zeta} = (\zeta_q) := (J'_q \mathcal{M}'_t m)$ and

$$(3.12) \quad (\bar{S}_t \bar{\mu})_q = \begin{cases} \sum_{z \in \Lambda} e^{\langle t, z-v \rangle} S'_z \mu_{q-z} + \sum_{\substack{p \in \Lambda_1 \\ z \in \Lambda}} e^{\langle t, z-v \rangle} J'_q \hat{S}'_z \mu_{p-z} & \forall q \notin \Lambda_1, \\ \sum_{z \in \Lambda} e^{\langle t, z-v \rangle} A'_z \mu_{q-z} + \sum_{\substack{p \in \Lambda_1 \\ z \in \Lambda}} e^{\langle t, z-v \rangle} J'_q \hat{S}'_z \mu_{p-z} & \forall q \in \Lambda_1. \end{cases}$$

A direct computation shows that, for each smooth local function φ , $\Pr(\mathbf{M}_t \boldsymbol{\mu})(\varphi) = \Pr(\boldsymbol{\mu})(\mathcal{M}_t \varphi)$, thus the lift covers the dynamics. In addition, one can easily check that $\mathbf{M}_0 = \mathbf{S}$ and that $\mathbf{M}_t$ is analytic in t .¹⁶ Accordingly, standard perturbation theory implies that there exists α_t , $\boldsymbol{\mu}_t$, analytic in t , such that $\mathbf{M}_t \boldsymbol{\mu}_t = \alpha_t \boldsymbol{\mu}_t$ with $\alpha_0 = 1$, $\boldsymbol{\mu}_0 = \boldsymbol{\mu}^w$.

We will normalize $\boldsymbol{\mu}_t$ so that $\boldsymbol{\mu}_t = (1, \bar{\mu}_t)$. Setting $\mu_t := \Pr(\boldsymbol{\mu}_t)$, for each fixed local function f , $\mu_t(f) = \Pr \boldsymbol{\mu}_t(f)$ is analytic in t since the sum implicit in the right hand side is just a finite sum.¹⁷ However Lemma 2.5 requires a more quantitative information.

By the arguments of section 3.2 (see (3.9)) it follows that $\mathbf{M}_0 = \mathbf{S} = \Pi + R$ where $\Pi^2 = \Pi$, $\Pi R = R \Pi = 0$ and $\|R^n\| \leq C\eta^n$, for all $n \in \mathbb{N}$. Thus by standard perturbation theory (see [7]), $\mathbf{M}_t = \alpha_t \Pi_t + R_t$ where $|\alpha_0 - \alpha_t| \leq C\|t\|$, $\|\Pi_t - \Pi\| \leq C\|t\|$, $\|R_t^n\| \leq C\eta_t^n$, with $\eta_t \leq \eta + C\|t\|$. Hence, $\|\mathbf{M}_t^n\| \leq C|\alpha_t|^n + C\eta_t^n$ and, for each local function f depending only on L variables

$$|\mathcal{M}_t^n \nu(f)| = |\Pr(\mathbf{M}_t^n \Psi(\nu))(f)| \leq L\|f\|_\infty \|\mathbf{M}_t^n \Psi(\nu)\| \leq C|\alpha_t|^n \|\nu\| L\|f\|_\infty$$

¹⁵This problem is already well investigated, see in particular [3], here we treat it in detail only because we need some explicit estimates not readily available in the literature.

¹⁶For the first assertion note that

$$\sum_{p \in \Lambda_1} \sum_{z \in \Lambda} \mu_{p-z}(\hat{S}_z 1) = \sum_{p \in \mathbb{Z}^d} \sum_{z \in \Lambda} \mu_{p-z}(\hat{S}_z 1) = \sum_{p \in \mathbb{Z}^d} \sum_{z \in \Lambda} \mu_p(S_z 1) = \sum_{p \in \mathbb{Z}^d} \mu_p(S 1) = 0.$$

For the latter just write it as power series of t .

¹⁷Note that μ_t is not necessarily a measure and gives rise to an analytic object only when applied to a local function. We will abuse notations by writing $\dot{\mu}_t$ to mean the functional on local functions defined by $\Pr(\frac{d}{dt} \boldsymbol{\mu}_t)(\varphi)$.

provided that $|\alpha_t| \geq \eta_t$ which holds for all $\|t\| < B$ for some $B > 0$. This proves the first inequality of Lemma 2.5.

To prove the second note that $\Pi_t \boldsymbol{\nu} = \ell_t(\boldsymbol{\nu}) \boldsymbol{\mu}_t$ with $\ell_0(\boldsymbol{\nu}) = [\boldsymbol{\nu}]_0$ hence¹⁸

$$\begin{aligned} \mathcal{M}_t^n \nu(1) &= [\mathbf{M}_t^n \Psi(\nu)]_0 = [\alpha_t^n \Pi_t \Psi(\nu) + \mathcal{O}(\eta_t^n \|\nu\|)]_0 \\ &= [\alpha_t^n \Pi_0 \Psi(\nu)]_0 + \mathcal{O}((\eta_t^n + \alpha_t^n C t) \|\nu\|) = \alpha_t^n (1 + \mathcal{O}(t \|\nu\|)) + \mathcal{O}(\eta_t^n \|\nu\|). \end{aligned}$$

Finally, to study the derivatives of α we use the relation $\mu_t(\mathcal{M}_t \varphi) = \alpha_t \mu_t(\varphi)$ for any local smooth function φ . Differentiating with respect to t yields

$$\begin{aligned} \dot{\mu}_t(\mathcal{M}_t \varphi) + \mu_t(\dot{\mathcal{M}}_t \varphi) &= \dot{\alpha}_t \mu_t(\varphi) + \alpha_t \dot{\mu}_t(\varphi) \\ (3.13) \quad \ddot{\mu}_t(\mathcal{M}_t \varphi) + 2\dot{\mu}_t(\dot{\mathcal{M}}_t \varphi) + \mu_t(\ddot{\mathcal{M}}_t \varphi) &= \ddot{\alpha}_t \mu_t(\varphi) + 2\dot{\alpha}_t \dot{\mu}_t(\varphi) + \alpha_t \ddot{\mu}_t(\varphi) \\ \dot{\mu}_t(1) &= \ddot{\mu}_t(1) = 0, \end{aligned}$$

Since $\dot{\mathcal{M}}_t = \sum_{z \in \Lambda} (z - v) e^{\langle t, z - v \rangle} S_z$ and $\ddot{\mathcal{M}}_t = \sum_{z \in \Lambda} (z - v) \otimes (z - v) e^{\langle t, z - v \rangle} S_z$ the above equations, for $t = 0$ imply (substituting $\varphi = 1$)

$$\dot{\alpha}_0 = \mu^w(\dot{\mathcal{M}}_0 1) = \mu^w(g - v) = 0,$$

where g is defined in (2.5) and we have used (2.6). Next, substituting in the first of the (3.13), $\varphi = \sum_{k=0}^{n-1} S^k \phi$, for some local function ϕ , we have

$$\sum_{k=0}^{n-1} \dot{\mathcal{M}}'_0 \mu^w(S^k \phi) = \dot{\mu}_0((1 - S) \sum_{k=0}^{n-1} S^k \phi) = \dot{\mu}_0(\phi) - \Pr(\mathbf{S}^{n-1} \dot{\boldsymbol{\mu}}_0)(\phi).$$

By (3.10), taking the limit for n to infinity, we have

$$(3.14) \quad \dot{\mu}_0(\phi) = \sum_{k=0}^{\infty} \dot{\mathcal{M}}'_0 \mu^w(S^k \phi).$$

Finally, the second of the (3.13), setting $\varphi = 1$ and $t = 0$, yields¹⁹

$$\begin{aligned} \ddot{\alpha}_0 &= 2 \sum_{n=0}^{\infty} \mu^w(\dot{\mathcal{M}}_0 S^n \dot{\mathcal{M}}_0 1) + \mu^w(\ddot{\mathcal{M}}_0 1) \\ (3.15) \quad &= 2 \sum_{n=0}^{\infty} \mathbb{E}_{\mu^w}(\tilde{\Delta}_n \otimes \tilde{\Delta}_0) + \mathbb{E}_{\mu^w}(\tilde{\Delta}_0 \otimes \tilde{\Delta}_0). \end{aligned}$$

Since $\dot{\mathcal{M}} 1$ is a local function the sum is convergent. Hence

$$\begin{aligned} \ddot{\alpha}_0 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[2 \sum_{k,m=0}^n \mathbb{E}_{\mu^w}(\tilde{\Delta}_{m+k} \otimes \tilde{\Delta}_k) + \mathbb{E}_{\mu^w}(\tilde{\Delta}_k \otimes \tilde{\Delta}_k) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mu^w}(\tilde{X}_n \otimes \tilde{X}_n) \geq 0. \end{aligned}$$

¹⁸Here we use the notation $[\boldsymbol{\nu}]_0$ to designate the components c_ν of the vector $\boldsymbol{\nu} = (c_\nu, \bar{\nu})$.

¹⁹Remember that $\ddot{\alpha}_t = (\partial_{t_i} \partial_{t_j} \alpha_t)$ is a $d \times d$ matrix.

Finally, if there exists $w \in \mathbb{R}^d$ such that $\ddot{\alpha}_0 w = 0$, it means (from (3.15) and Theorem 2.2) that there exists a constant $C_5 > 0$ such that, for all $n \in \mathbb{N}$,²⁰

$$\mathbb{E}_{\mu^w} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) \leq C_5 ; \quad \mathbb{E} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) \leq C_5$$

We can thus extract a subsequence $\{n_j\}$ such that $\langle w, \tilde{X}_{n_j} \rangle$ converges weakly almost surely to a random variable Z . Let $\psi = \mathbb{E}_{\mu^w}(Z \mid \mathcal{F}_0)$ and $\tilde{g}_w = \langle w, g - v \rangle$, then, for each $\mathcal{F}_0$ measurable smooth local function φ ,

$$\begin{aligned} \mathbb{E}_{\mu^w}(\varphi(\psi - S\psi)) &= \lim_{j \rightarrow \infty} \mathbb{E}_{\mu^w}(\varphi(\tilde{X}_{n_j} - \tilde{X}_{n_j+1})) = \lim_{j \rightarrow \infty} \mu^w(\varphi(\tilde{g}_w - S^{n_j+1}\tilde{g}_w)) \\ &= \mu^w(\varphi\tilde{g}_w), \end{aligned}$$

where we have used Theorem 2.2. Thus $\tilde{g}_w = \psi - S\psi$, μ^e -a.s.. This implies that, setting $M_0 = 0$, and

$$\begin{aligned} M_{n+1} - M_n &= \langle w, \tilde{\Delta}_n \rangle - \mathbb{E}(\langle w, \tilde{\Delta}_n \rangle \mid \mathcal{F}_n) + \psi(\omega^{n+1}) - S\psi(\omega^n) \\ &= \langle w, \tilde{\Delta}_n \rangle + \psi(\omega^{n+1}) - \psi(\omega^n), \end{aligned}$$

M_n is a $\mathbb{P}_{\mu^w}$ stationary martingale. Moreover,

$$\langle w, \tilde{X}_n \rangle = M_n - \psi(\omega^n) + \psi(\omega^0).$$

From this it follows that

$$\begin{aligned} C &\geq \mathbb{E}_{\mu^w}(|M_n|^2) = \sum_{k=1}^{n-1} \mathbb{E}_{\mu^w}(|\langle w, \tilde{\Delta}_n \rangle + \psi(\omega^{n+1}) - \psi(\omega^n)|^2) \\ &= \sum_{k=1}^{n-1} \mathbb{E}_{\mu^w}(|\langle w, \tilde{\Delta}_n \rangle + \psi(\omega^{n+1})|^2 - |\psi(\omega^n)|^2) \\ &= (n-1) \left[\mathbb{E}_{\mu^w}(|\langle w, \tilde{\Delta}_1 \rangle + \psi(\omega^1)|^2 - |\tilde{g}_w(\omega^0) + S\psi(\omega^0)|^2) \right]. \end{aligned}$$

Thus $\sum_z \pi_z |\langle w, z - v \rangle + \psi \circ F \circ \tau^z|^2 = |\sum_z \pi_z (\langle w, z - v \rangle + \psi \circ F \circ \tau^z)|^2$, that is $\langle w, z - v \rangle + \psi \circ F \circ \tau^z = \tilde{g}_w + S\psi = \psi$, μ^w -a.s..

Next, let $\alpha_z = \mu^w(\pi_z)$, then, $\sum_z \alpha_z = 1$ and $\sum_z \langle w, z - v \rangle \alpha_z = \mu^w(\tilde{g}_w) = 0$. Hence,

$$\sum_z \alpha_z \psi \circ F \circ \tau^z = \psi \quad \mu^w\text{-a.s..}$$

²⁰The latter follows by Theorem 2.2. Let $\beta = \sum_z \langle w, z - v \rangle^2 \pi_z$, and $G_j = \sum_z \langle w, z - v \rangle S_z(S^j \langle w, g - v \rangle)$, then

$$\mathbb{E} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) = \sum_{k=0}^{n-1} \mu^e(S^k \beta) + \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \mu^e(S^k G_j)$$

and $|\mu^e(S^k \beta) - \mu^w(\beta)| \leq C\eta^k$, $|\mu^e(S^k G_j) - \mu^w(G_j)| \leq Cj^d \eta^k$. Moreover, setting $\nu_{k,z}(\varphi) := \langle w, z - v \rangle \mu^e(S^k S_z \varphi)$, for ε in assumption 3 such that $(1 + 3\varepsilon)2\lambda^{-1} < 1$ we have (by equation (3.6) and (3.7)) $\|\nu_{k,z}\| \leq C$. Hence Theorem 2.2 yields

$$|\mu^e(S^k G_j)| \leq \left| \sum_z \nu_{k,z}(S^j(g - v)) \right| \leq C\eta^j$$

Thus we can write

$$\left| \mathbb{E} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) - \mathbb{E}_{\mu^w} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) \right| \leq C \sum_{k=0}^n \eta^k + C \sum_{j=1}^{n-1} \left[\sum_{k=0}^j \eta^k + \sum_{k=j+1}^{n-j} j^d \eta^k \right] \leq C.$$

Note that the operator $S_\alpha \varphi := \sum_z \alpha_z \varphi \circ F \circ \tau^z$ defines a Markov process with invariant measure μ^e and satisfies the hypothesis of Theorem 2.2. Since $\psi = \sum_{k=0}^{n-1} S^k \tilde{g}_w + S^n \psi$, for each $\phi \in L^2(\mu^w)$,

$$\lim_{j \rightarrow \infty} \mu^w(\phi \sum_{k=0}^{n_j-1} S^k \tilde{g}_w) = \lim_{j \rightarrow \infty} \mathbb{E}_{\mu^w}(\phi(\omega^0) \langle w, X_{n_j} \rangle) = \mu^w(\phi \psi).$$

In addition, assumption 3 implies that setting,²¹ for each smooth local function ϕ , $\nu_{n,\phi}(\varphi) = \mu^w(\phi S_\alpha^n \varphi)$,

$$|\nu_{n,\phi}(\varphi)| \leq \frac{(1+\varepsilon)^n}{(1-\varepsilon)^n} |\phi|_\infty \mu^w(S^n |\varphi|) = \frac{(1+\varepsilon)^n}{(1-\varepsilon)^n} |\phi|_\infty \mu^w(|\varphi|).$$

Thus $\nu_{n,\phi}$ is absolutely continuous with respect to μ^w with density $\rho_{n,\phi} \in L^\infty(\mu^w)$. Accordingly,²²

$$\begin{aligned} \mu^w(\phi \psi) &= \mu^w(\phi S_\alpha^n \psi) = \lim_{j \rightarrow \infty} \sum_{k=0}^{n_j-1} \mu^w(\phi S_\alpha^n S^k \tilde{g}_w) \\ &= \sum_{k=0}^{n_l-1} [\mu^e(S^k \tilde{g}_w) \mu^w(\phi) + \mathcal{O}(C_\phi \eta^n k^d)] + \lim_{j \rightarrow \infty} \sum_{k=n_l}^{n_j-1} \mathcal{O}(C_\phi \eta^k) \\ &= \sum_{k=0}^{n_l-1} \mu^e(S^k \tilde{g}_w) \mu^w(\phi) + C_\phi \mathcal{O}(\eta^{n_l} + \eta^n n_l^d). \end{aligned}$$

Taking first the limit for $n \rightarrow \infty$ and the one $l \rightarrow \infty$ yields $\mu^w(\phi \psi) = \mu^w(\phi) \mu^e(\psi)$. That is ψ is μ^w almost surely constant. This implies that $\tilde{g}_w = 0$ and hence $\langle w, z - v \rangle \pi_z = 0$, μ^w a.s.. This is equivalent to saying that the vectors in the set $\{(\langle e_1, z \rangle, \dots, \langle e_d, z \rangle)\}_{z \in \Lambda} \cup \{(1, \dots, 1)\}$ are linearly dependent over $\mathbb{R}$, but this implies that they are linearly dependent over $\mathbb{Z}$. In other words we can assume that $w \in \mathbb{Z}^d$. Finally, since π_z is smooth, we have $\langle w, z \rangle = \langle w, v \rangle$ unless $\pi_z \equiv 0$, which contradicts Assumption 4. $\square$

3.4. Variation bounds for conditional measures. In the previous subsection we obtained several results for random walks provided that we start the environment in a measure with “density” of bounded variation. Here we show why such measures constitute a natural class for the problem at hand. More precisely we shall show that if we start with a nice measure and condition on a behavior of a walk during an initial time interval we still have a good control on the variation of densities.

For future needs we consider two random walks (X_t, Y_t) evolving in the same environment starting respectively at $a, b \in \mathbb{Z}^d$ and with the environment at time zero distributed according to the measure $\nu \in \mathcal{B}$. Let $\mathbb{P}_{a,b,\nu}^2$ be the measure on $(\Theta \times \mathbb{Z}^{2d})^\mathbb{N}$ associated to such a process and $\mathbb{E}_{a,b,\nu}^2$ the corresponding expectation $\mathbb{P}_\nu^2 := \mathbb{P}_{0,0,\nu}^2$ and $\mathbb{E}_\nu^2 := \mathbb{E}_{0,0,\nu}^2$.

Let $m \in \mathbb{N}$ and consider the σ -algebra $\mathcal{F}_m^{XY} = \sigma\{X_1, Y_1, \dots, X_m, Y_m\}$. We are interested in computing $\mathbb{E}_{a,b,\nu}^2(f(X, Y, \theta^m) \mid \mathcal{F}_m^{XY})$ for each local $\mathcal{F}_m^{XY} \otimes \mathcal{T}$ -measurable function f and probability measure $\nu \in \mathcal{B}$. Thus, we are interested in

²¹Indeed, $\mu^w(\pi_z) \leq a_z(1+\varepsilon)$, thus $\pi_z \geq (1-\varepsilon)a_z \geq (1-\varepsilon)(1+\varepsilon)^{-1}a_z$.

²²Note that (3.6) imply $\|S'_\alpha \mu\| \leq \sum_z \alpha_z \|F' \mu\| \leq (2\lambda)^{-1} \|\mu\| + B\|\mu\|$ and $|S_\alpha \mu| \leq |\mu|$. Thus, iterating, for each $n \in \mathbb{N}$, $\|(S_\alpha^n)' \mu\| \leq C\|\mu\|$.

the measures ν_m^{XY} defined by

$$\mathbb{E}_{a,b,\nu}^2(f(X, Y, \theta^m) \mid \mathcal{F}_m^{XY}) =: \int_{\Theta} f(X, Y, \theta) \nu_{a,b,m}^{XY}(d\theta).$$

Lemma 3.5. *There exists $C_6 > 0$ and $0 < \varepsilon_1 \leq \varepsilon_0$ such that, if assumption 3 is satisfied for ε_1 , then for each $m \in \mathbb{N}$, $a, b \in \mathbb{Z}^d$ and probability measure $\nu \in \mathcal{B}$ the following holds*

$$\|\nu_{a,b,m}^{XY}\| \leq C_6 \|\nu\|.$$

Proof. Given two random walks realizations $X, Y : \mathbb{N} \rightarrow \mathbb{R}^d$, let us define the operators

$$S_{X,Y,k} f(\theta) := \pi_{z_k}(\tau^{X_k} \theta) \pi_{w_k}(\tau^{Y_k} \theta) f \circ F(\theta),$$

where $z_k = X_{k+1} - X_k$ and $w_k = Y_{k+1} - Y_k$. With such a notation we can write

$$\nu_{a,b,m}^{XY} = \frac{S'_{X,Y,m-1} \cdots S'_{X,Y,0} \nu}{S'_{X,Y,m-1} \cdots S'_{X,Y,0} \nu(1)}.$$

Recalling (3.7) and the Lasota-Yorke inequality for the map F (see (3.6)), and using Assumption 3 we have

$$(3.16) \quad \|S'_{X,Y,k} \nu\| \leq 2\lambda^{-1}(1 + \varepsilon_1)^2 \|\nu\| a_{z_k} a_{w_k} + B S'_{X,Y,k} \nu(1).$$

Hence, for ε_1 such that $2\lambda^{-1}(1 + \varepsilon_1)^2 \leq \eta(1 - \varepsilon_1)^{-2} < 1$ we can iterate the above inequality and obtain

$$\begin{aligned} \|S'_{X,Y,m-1} \cdots S'_{X,Y,0} \nu\| &\leq \eta^m (1 - \varepsilon_1)^{2m} \|\nu\| \prod_{k=0}^{m-1} a_{z_k} a_{w_k} \\ &\quad + B \sum_{j=0}^{m-1} \eta^j (1 - \varepsilon_1)^{2j} \nu(S_{X,Y,0} \cdots S_{X,Y,m-1-j} 1) \prod_{k=m-j}^{m-1} a_{z_k} a_{w_k} \\ &\leq [\eta^m \|\nu\| + (1 - \eta)^{-1} B] \nu(S_{X,Y,0} \cdots S_{X,Y,m-1} 1), \end{aligned}$$

which proves the Lemma with $C_6 = 1 + (1 - \eta)^{-1} B$.²³ $\square$

4. ANNEALED INVARIANCE PRINCIPLE.

This section is devoted to proving an averaged invariance principle. This result is used in Section 5 to prove Lemma 2.8.

Consider the process

$$(4.1) \quad \hat{X}_t^N = \frac{1}{\sqrt{N}} \left\{ \tilde{X}_{\lceil tN \rceil} + (tN - \lceil tN \rceil) \tilde{\Delta}_{\lceil tN \rceil} \right\}.$$

Note that $\hat{X}_t^N \in \mathcal{C}^0([0, 1], \mathbb{R}^d)$, by construction. In fact, Lemma 2.6 implies higher regularity.

Lemma 4.1. *The family of processes $\{\hat{X}^N\} \subset \mathcal{C}^0([0, 1], \mathbb{R}^d)$ is tight.*

Proof. Let $\varsigma \in (0, 1/2)$,

$$L_{\varsigma}(f) := \sup_{t,s \in [0,1]} \frac{\|f(t) - f(s)\|}{|t - s|^{\varsigma}},$$

and $K_L^{\varsigma} := \{f \in \mathcal{C}^0([0, 1], \mathbb{R}^d) : f(0) = 0, L_{\varsigma}(f) \leq L\}$.

²³Note that, for a probability measure, $1 = \nu(1) = \nu(\partial_{\theta_i} \theta_i) \leq \|\nu\|$.

By Lemma 2.6 it follows that, for each $N \in \mathbb{N}$, $t \in [0, 1]$ and $h \in [-t, 1 - t]$,

$$(4.2) \quad \mathbb{P} \left(\left\{ \|\hat{X}_{t+h}^N - \hat{X}_t^N\| \geq Lh^\varsigma \right\} \right) \leq e^{-CL^2 h^{2\varsigma-1}}.$$

In addition, if $\|\hat{X}_t^N\| + \|\hat{X}_{t+h}^N\| \leq L^{1-\varsigma}$, then the set in (4.2) is empty for all $h > L^{-1}$. Now the result follows in complete analogy with the usual proof of the Hölder continuity of the Brownian motion, based on applying the above estimates to the dyadic rationals, yielding

$$\mathbb{P} \left(\left\{ \hat{X}^N \notin K_L^\varsigma \right\} \right) \leq e^{-CL}.$$

Since K_L^ς are compact in $\mathcal{C}^0([0, 1], \mathbb{R}^d)$ the tightness follows. $\square$

Lemma 4.1 also allows us to prove the invariance principle.

Lemma 4.2. *For each probability measure $\nu \in \mathcal{B}$ the process $\{\hat{X}_t^N\}$ converges in law to the Brownian motion with diffusion matrix Σ^2 .*

Proof. In view of Lemma 4.1 we only need to check the convergence of finite dimensional distributions. We consider two dimensional distributions, the general case being very similar. Accordingly, let $t_1 < t_2$ and fix ξ_1, ξ_2 . We have

$$\begin{aligned} \mathbb{E}_\nu \left(\exp(i\langle \xi_1, \hat{X}_{t_1}^N \rangle + i\langle \xi_2, \hat{X}_{t_2}^N \rangle) \right) &= \mathbb{E}_\nu \left(\exp(i\langle [\xi_1 + \xi_2], \hat{X}_{t_1}^N \rangle) \exp(i\langle \xi_2, [\hat{X}_{t_2}^N - \hat{X}_{t_1}^N] \rangle) \right) = \\ &= \mathbb{E}_\nu \left(\mathbb{E} \left(\exp(i\langle \xi_2, [\hat{X}_{t_2}^N - \hat{X}_{t_1}^N] \rangle) \middle| \mathcal{F}_{[t_1 N]} \right) \exp(i\langle [\xi_1 + \xi_2], \hat{X}_{t_1}^N \rangle) \right). \end{aligned}$$

By Lemma 3.5 and Proposition 2.4 we have²⁴

$$\mathbb{E} \left(\exp(i\langle \xi_2, [\hat{X}_{t_2}^N - \hat{X}_{t_1}^N] \rangle) \middle| \mathcal{F}_{[t_1 N]} \right) = \exp \left(-\frac{1}{2} \langle \xi_2, \Sigma^2 \xi_2 \rangle (t_2 - t_1) (1 + o(1)) \right)$$

and so using Proposition 2.4 again we obtain

$$\mathbb{E}_\nu \left(\exp(i\langle \xi_1, \hat{X}_{t_1}^N \rangle + i\langle \xi_2, \hat{X}_{t_2}^N \rangle) \right) \sim e^{-\frac{1}{2} [\langle \xi_2, \Sigma^2 \xi_2 \rangle (t_2 - t_1) + \langle (\xi_1 + \xi_2), \Sigma^2 (\xi_1 + \xi_2) \rangle t_1]}$$

Thus, $(\hat{X}_{t_1}^N, \hat{X}_{t_2}^N)$ is asymptotically Gaussian with zero mean and the variance predicted by the Brownian Motion. $\square$

²⁴In fact, Lemma 3.5 considers two walks, yet the corresponding result for one walk can be obtained by integrating over the second walk. Moreover, for each smooth function $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathbb{E}_\nu (f(\hat{X}_{t_2}^N, \hat{X}_{t_1}^N) \mid \mathcal{F}_{[t_1 N]}) &= \mathbb{E}_\nu \left(\mathbb{E} (f(\hat{X}_{t_2}^N, \hat{X}_{t_1}^N) \mid \tilde{X}_{[t_1 N]}, \theta^{[t_1 N]}) \mid \mathcal{F}_{[t_1 N]} \right) \\ &= \nu_{\tilde{X}_{[t_1 N]}, [t_1 N]}^X \left(\mathbb{E} (f(\hat{X}_{t_2}^N, \hat{X}_{t_1}^N) \mid \tilde{X}_{[t_1 N]}, \theta^{[t_1 N]}) \right) \\ &= \mathbb{E}_{X_{t_1}^N, \nu_{\tilde{X}_{[t_1 N]}, [t_1 N]}^X} \left(\mathbb{E} (f(\hat{X}_{t_2}^N - \hat{X}_0^N, \hat{X}_0^N) \mid \tilde{X}_0, \theta^0) \right) = \mathbb{E}_{X_{t_1}^N, \nu_{\tilde{X}_{[t_1 N]}, [t_1 N]}^X} \left(f(\hat{X}_{t_2}^N - \hat{X}_0^N, \hat{X}_0^N) \right). \end{aligned}$$

Proposition 2.4 can then be applied after translating $\nu_{\tilde{X}_{[t_1 N]}, [t_1 N]}^X$ by $\tilde{X}_{[t_1 N]}$.

5. PROOFS: QUENCHED CLT VIA THE STUDY OF TWO RANDOM WALKS

The goal of this section is to establish Lemma 2.8.

Lemma 4.1 shows that the distributions of the processes $(\hat{X}_t^N, \hat{Y}_t^N)$ are tight, hence they have accumulation points. Our next task is to characterize such accumulation points. Let us consider any accumulation point $(\hat{X}_t^\infty, \hat{Y}_t^\infty)$. We will see that $(\hat{X}_t^\infty, \hat{Y}_t^\infty)$ is a centered Gaussian random variables with variance

$$(5.1) \quad \Sigma_2^2 := t \begin{pmatrix} \Sigma^2 & 0 \\ 0 & \Sigma^2 \end{pmatrix}$$

More precisely, if we define the second order differential operator $\Delta_{\Sigma_2^2} := \sum_{i,j} \Sigma_{2ij}^2 \partial_i \partial_j$ we have the following.

Proposition 5.1. *For any $\psi \in \mathcal{C}^3(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ we have*

$$\frac{d}{dt} \mathbb{E}^2(\psi(t, \hat{X}_t^\infty, \hat{Y}_t^\infty)) = \mathbb{E}^2(\partial_t \psi(t, \hat{X}_t^\infty, \hat{Y}_t^\infty) + \frac{1}{2} \Delta_{\Sigma_2^2} \psi(t, \hat{X}_t^\infty, \hat{Y}_t^\infty)).$$

More precisely, there exists $\beta \in (0, \frac{1}{4})$ and $\vartheta \in (2\beta, 1 - 2\beta)$ such that, for all $N \in \mathbb{N}$ and $t, h \in [0, 1]$ such that $h > N^{\vartheta-1}$ we have

$$\begin{aligned} & \left| \mathbb{E}^2 \left(\psi(t+h, \hat{X}_{t+h}^N, \hat{Y}_{t+h}^N) - \psi(t, \hat{X}_t^N, \hat{Y}_t^N) - h \left[\partial_t \psi(t, \hat{X}_t^N, \hat{Y}_t^N) + \frac{1}{2} \Delta_{\Sigma_2^2} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \right] \right) \right| \\ & \leq C \|\psi\|_{\mathcal{C}^3} (N^{-\beta} h + h^{\frac{3}{2}} + N^{-\frac{1}{2}}). \end{aligned}$$

Thanks to the above Proposition for each $\phi \in \mathcal{C}_0^3(\mathbb{R}^d \times \mathbb{R}^d)$ we can define ψ by

$$(5.2) \quad \begin{aligned} \partial_t \psi + \frac{1}{2} \Delta_{\Sigma_2^2} \psi &= 0 \\ \psi(1, x, y) &= \phi(x, y) \end{aligned}$$

and, by applying Proposition 5.1 with the choice $h = \lceil N^{2\beta} \rceil^{-1}$, obtain the wanted result:

$$\begin{aligned} \mathbb{E}^2(\phi(\hat{X}_1^N, \hat{Y}_1^N)) &= \mathbb{E}^2(\psi(1, \hat{X}_1^N, \hat{Y}_1^N)) \\ &= \mathbb{E}^2(\psi(0, \hat{X}_0^N, \hat{Y}_0^N)) + \sum_{i=0}^{h^{-1}-1} \mathbb{E}^2(\psi((i+1)h, \hat{X}_{(i+1)h}^N, \hat{Y}_{(i+1)h}^N) - \psi(ih, \hat{X}_{ih}^N, \hat{Y}_{ih}^N)) \\ &= \mathbb{E}^2(\psi(0, 0, 0)) + \mathcal{O}(\|\psi\|_{\mathcal{C}^3} N^{-\beta}) = \mathbb{E}_{\mathcal{N}(0, \Sigma_2^2)}(\phi) + \mathcal{O}(\|\phi\|_{\mathcal{C}^3} N^{-\beta} + N^{-\frac{1}{2}+2\beta}), \end{aligned}$$

where we have used the explicit solution of (5.2).²⁵ Remembering the form of Σ_2^2 (see (5.1)), Lemma 2.8, and hence Theorem 2.7, follow.

²⁵Indeed, equation (5.2) is just the backward heat equation, thus for $t \in (0, 1)$

$$\psi(t, x, y) = \frac{1}{(4\pi)^d \det(\Sigma_2^{-1})(1-t)^d} \int_{\mathbb{R}^{2d}} e^{-\frac{\langle (x-z, y-w), \Sigma_2^{-2}(x-z, y-w) \rangle}{2(1-t)}} \phi(z, w) \, dz \, dw.$$

Proof of Proposition 5.1. We start by the following Taylor expansion

$$\begin{aligned}
& \mathbb{E}^2(\psi(t+h, \hat{X}_{t+h}^N, \hat{Y}_{t+h}^N)) - \mathbb{E}^2(\psi(t, \hat{X}_t^N, \hat{Y}_t^N)) = \mathbb{E}^2(\partial_t \psi(t, \hat{X}_t^N, \hat{Y}_t^N))h \\
& + \mathbb{E}^2(\partial_x \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N) + \partial_y \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N)) \\
& + \frac{1}{2} \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N) \cdot \partial_x^2 \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N)) \\
& + \mathbb{E}^2((\hat{Y}_{t+h}^N - \hat{Y}_t^N) \cdot \partial_{xy} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N)) \\
(5.3) \quad & + \frac{1}{2} \mathbb{E}^2((\hat{Y}_{t+h}^N - \hat{Y}_t^N) \cdot \partial_y^2 \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N)) \\
& + \frac{1}{2} \mathbb{E}^2(\partial_{tx} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N))h \\
& + \frac{1}{2} \mathbb{E}^2(\partial_{ty} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N))h \\
& + \mathcal{O}\left(|\partial_t^2 \psi|_\infty h^2 + \|\psi\|_{C^3} \left[\mathbb{E}(\|\hat{X}_{t+h}^N - \hat{X}_t^N\|^3) + \mathbb{E}(\|\hat{Y}_{t+h}^N - \hat{Y}_t^N\|^3) \right]\right).
\end{aligned}$$

Next, we will analyze the terms in equation (5.3) one by one.

First of all note that the remainders are of order $h^{\frac{3}{2}}$.²⁶ Let us start the estimates with the term $\mathbb{E}(\partial_x \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N))$. To this end we consider the σ -algebra $\mathcal{F}_m^{XY}$ generated by $\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$. Setting $\tilde{g} = g - v$, $\ell_t := \lceil tN \rceil + 1$ we can write

$$\begin{aligned}
\mathbb{E}((\hat{X}_{t+h}^N - \hat{X}_t^N) \mid \mathcal{F}_{\ell_t}^{XY}) &= N^{-\frac{1}{2}} \sum_{k=\ell_t}^{\ell_{t+h}} \mathbb{E}(\tilde{\Delta}_k^X \mid \mathcal{F}_{\ell_t}^{XY}) + \mathcal{O}(N^{-\frac{1}{2}}) \\
&= N^{-\frac{1}{2}} \sum_{k=\ell_t}^{\ell_{t+h}} \mathbb{E}((S^{k-\ell_t} \tilde{g}) \circ \tau^{X_{\ell_t}} \mid \mathcal{F}_{\ell_t}^{XY}) + \mathcal{O}(N^{-\frac{1}{2}}) \\
&= N^{-\frac{1}{2}} \sum_{k=\ell_t}^{\ell_{t+h}} [(S^{k-\ell_t} \tau^{X_{\ell_t}})'(\mu^e)_{\ell_t}^{XY}](\tilde{g}) + \mathcal{O}(N^{-\frac{1}{2}}).
\end{aligned}$$

where, by Lemma 3.5, $\|(\mu^e)_{\ell_t}^{XY}\| \leq C$ and hence $\|(\tau^{X_{\ell_t}})'(\mu^e)_{\ell_t}^{XY}\| \leq C$. From Theorem 2.2 and the fact that $\mu^w(\tilde{g}) = 0$ it follows that

$$(5.4) \quad |\mathbb{E}^2(\partial_x \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N))| \leq CN^{-\frac{1}{2}} \|\psi\|_{C^1}.$$

In complete analogy we have

$$(5.5) \quad |\mathbb{E}^2(\partial_y \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N))| \leq CN^{-\frac{1}{2}} \|\psi\|_{C^1}.$$

The quadratic terms involving ∂_{tx} and ∂_{ty} are estimated in the same manner yielding terms of order $N^{-\frac{1}{2}} h \|\psi\|_{C^2}$. The quadratic terms involving only X or only Y

²⁶ In fact Lemma 2.6 implies, for each $p \in \mathbb{N}$, that

$$\begin{aligned}
\mathbb{E}(\|\hat{X}_{t+h}^N - \hat{X}_t^N\|^p) &\leq N^{\frac{p}{2}} h^p \mathbb{E}\left(\left\|\frac{1}{\lceil hN \rceil} (\tilde{X}_{\lceil (t+h)N \rceil} - \tilde{X}_{\lceil tN \rceil})\right\|^p\right) \\
&\leq Ch^{\frac{p}{2}} + C_p N^{\frac{p}{2}} h^p \int_{(Nh)^{-\frac{1}{2}}}^{\infty} x^{p-1} \mathbb{P}\left(\left\|\frac{1}{\lceil hN \rceil} (\tilde{X}_{\lceil (t+h)N \rceil} - \tilde{X}_{\lceil tN \rceil})\right\| \geq x\right) dx \\
&\leq Ch^{\frac{p}{2}} + C_p N^{\frac{p}{2}} h^p \int_{(Nh)^{-\frac{1}{2}}}^{\infty} x^{p-1} e^{-Cx^2 hN} dx \leq C_p h^{\frac{p}{2}}.
\end{aligned}$$

yield the following

(5.6)

$$\begin{aligned}\mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i(\hat{X}_{t+h}^N - \hat{X}_t^N)_j \partial_{x_i x_j} \psi) &= (\Sigma^2)_{ij} \mathbb{E}^2(\partial_{x_i x_j} \psi) h + \mathcal{O}\left(\frac{\|\psi\|_{C^2}}{\sqrt{N}}\right) \\ \mathbb{E}^2((\hat{Y}_{t+h}^N - \hat{Y}_t^N)_i(\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j \partial_{y_i y_j} \psi) &= (\Sigma^2)_{ij} \mathbb{E}^2(\partial_{y_i y_j} \psi) h + \mathcal{O}\left(\frac{\|\psi\|_{C^2}}{\sqrt{N}}\right).\end{aligned}$$

Indeed, in analogy with what we have done before, we can condition with respect to $\mathcal{F}_{\ell_t}^{XY}$ and

(5.7)

$$\begin{aligned}\mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i(\hat{X}_{t+h}^N - \hat{X}_t^N)_j \mid \mathcal{F}_{\ell_t}^{XY}) &= \frac{1}{N} \left\{ \sum_{k=\ell_t}^{\ell_{t+h}-1} \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_k^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right. \\ &\quad \left. + \sum_{k=\ell_t}^{\ell_{t+h}-1} \sum_{l=\ell_t}^{k-1} \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_l^X)_j + (\tilde{\Delta}_l^X)_i(\tilde{\Delta}_k^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right\} \\ &\quad + N^{-1} \mathcal{O} \left(1 + \sum_{l=\ell_t+1}^{\ell_{t+h}-1} \mathbb{E}^2((\tilde{\Delta}_{\ell_{t+h}-1}^X)_i(\tilde{\Delta}_l^X)_j + (\tilde{\Delta}_{\ell_{t+h}-1}^X)_j(\tilde{\Delta}_l^X)_i \mid \mathcal{F}_{\ell_t}^{XY}) \right),\end{aligned}$$

where the boundary terms of the type $\mathbb{E}^2((\tilde{\Delta}_{\ell_{t+h}-1}^X)_i(\tilde{\Delta}_l^X)_j \mid \mathcal{F}_{\ell_t}^{XY})$ have been estimated as in (5.4). To estimate such an expression note that

$$\mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_l^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) = (S^{j-\ell_t} \tau^{X_{\ell_t}})' \nu_{\ell_t}^{XY} (\tilde{g}_j S^{k-j} \tilde{g}_k)$$

Since Lemma 3.5 and Theorem 2.2 imply $\|(S^{j-\ell_t} \tau^{X_{\ell_t}})' \nu_{\ell_t}^{XY}\| \leq C \|\mu^e\|$ and also that $\|\tilde{g}_j (S^{j-\ell_t} \tau^{X_{\ell_t}})' \nu_{\ell_t}^{XY}\| \leq C \|\mu^e\|$ we can apply Theorem 2.2

$$\begin{aligned}\left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_l^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right| &= \nu_{\ell_t}^{XY} (S^{j-\ell_t} \tau^{X_{\ell_t}} \tilde{g}_j) \mu^w(\tilde{g}_k) + \mathcal{O}(\eta^{k-j}) = \mathcal{O}(\eta^{k-j}) \\ \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_l^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right| &= \nu_{\ell_t}^{XY} (1) \mu^w(\tilde{g}_j S^{k-j} \tilde{g}_k) + \mathcal{O}(\eta^{j-\ell_t}) \\ &= \mu^w(\tilde{g}_j S^{k-j} \tilde{g}_k) + \mathcal{O}(\eta^{j-\ell_t}).\end{aligned}$$

Using such estimates in (5.7) and remembering formula (3.15) (for $\ddot{\alpha}_0 = \Sigma^2$) the first equation of (5.6) follows. The second equation of (5.6) is proven in complete analogy.

Finally, we must deal with the mixed quadratic term.

$$\left| \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i(\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right| \leq N^{-1} \sum_{k,m=\ell_t-1}^{\ell_{t+h}} \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right|.$$

If $|k-m| \geq A \ln N$, then, for $A > \ln \eta^{-1}$, by Lemma 3.5 and Theorem 2.2 it follows that

$$\begin{aligned}\left| \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i(\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right| &= \sum_{\substack{\ell_t + 2A \ln N \leq k \leq \ell_{t+h} \\ |m-k| \leq A \ln N}} \frac{\mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{\ell_t}^{XY})}{N} \\ &\quad + \mathcal{O}(N^{-\frac{1}{2}}).\end{aligned}$$

Next, suppose that $|m-k| \leq A \ln N$ and $|X_{k-A \ln N} - Y_{k-A \ln N}| > 4C_0 A \ln N$.

Assume, to fix our ideas, that $k \leq m$ then for all times l such that $|l-k| \leq A \ln N$ the two walks explore disjoint parts of the environment. Thus we can consider the process started at time $k - A \ln N$ with the conditional measure $(\mu^e)_{k-A \ln N}^{XY}$ and with the walks starting from $a = X_{k-A \ln N}, b = Y_{k-A \ln N}, \|a-b\| > 4C_0 A \ln N$. If we set $\mathbf{f} = S^{A \ln N} \tau^{X_k} \tilde{g}_i$ and $\mathbf{h} = S^{m-k+A \ln N} \tau^{Y_k} \tilde{g}_j$ we have that the two functions depend on different sets of variables (let $B \subset \mathbb{Z}^d$ be the set of variables on which $\mathbf{f}$ depends and B' the ones relative to $\mathbf{h}$) and

$$\mathbb{E}^2((\tilde{\Delta}_k^X)_i (\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{k-A \ln N}^{XY}) = (\mu^e)_{k-A \ln N}^{XY}(\mathbf{f} \mathbf{h}).$$

We can then define its Newtonian potential of $\mathbf{f}$,

$$\begin{aligned} \Psi(\theta) &= \frac{1}{|B|(|B|-2)\alpha_{|B|}} \int_{I^{\mathbb{Z}^d}} \|\theta^B - \vartheta^B\|^{-|B|+2} \mathbf{f}(\vartheta) d\vartheta^B \otimes_{j \notin B} \mu_0(d\vartheta_j) \\ (5.8) \quad &= \frac{1}{|B|(|B|-2)\alpha_{|B|}} \int_{I^B} \|\theta^B - \vartheta^B\|^{-|B|+2} \mathbf{f}(\vartheta^B) d\vartheta^B. \end{aligned}$$

where $\theta^B = (\theta_l)_{l \in B}$ and α_l is the volume of the unit ball in $\mathbb{R}^l$. It is well known that, for θ^B in the interior of I^B

$$\sum_{l \in B} \partial_{\theta_l} \Psi = \mathbf{f}.$$

Thus, remembering Lemma 3.5, we can write²⁷

$$\begin{aligned} \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i (\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{k-A \ln N}^{XY}) \right| &\leq \sum_{l \in B} |(\mu^e)_{k-A \ln N}^{XY}(\partial_{\theta_l}(\partial_{\theta_l} \Psi \cdot \mathbf{h}))| \\ &\leq |B| \cdot \|(\mu^e)_{k-A \ln N}^{XY}\| \cdot \sup_{l \in B} |\partial_{\theta_l} \Psi \cdot \mathbf{h}|_{\infty} \leq C A^d C_0^d |\ln N|^d \cdot |g|_{\infty} \cdot \sup_{l \in B} |\partial_{\theta_l} \Psi|_{\infty}. \end{aligned}$$

By (5.8) we have, for $l \in B$,

$$\partial_{\theta_l} \Psi(\theta) = \int_{I^{\mathbb{Z}^d}} \frac{\theta_l - \vartheta_l}{|B|(|B|-2)\alpha_{|B|} \cdot \|\theta^B - \vartheta^B\|^{|B|}} \mathbf{f}(\vartheta) d\vartheta^B \otimes_{j \notin B} \mu_0(d\vartheta_j) =: \nu_l^{\theta}(\mathbf{f}).$$

Unfortunately, $\nu_l^{\theta} \notin \mathcal{B}$ due to the singularity of the kernel. To take care of this problem we need to isolate the singularity. For each $r > 0$ let $\chi_r \in \mathcal{C}^{\infty}(\mathbb{R}^B, [0, 1])$ such that $\chi_r(\theta^B) = 0$ for all $\|\theta^B\| \leq r$ and $\chi_r(\theta^B) = 1$ for all $\|\theta^B\| \geq 2r$. Clearly χ_r can be chosen radial and so that $\sup_l |\partial_{\theta_l} \chi_r|_{\infty} \leq C r^{-1}$. We then define

$$\nu_{l,r}^{\theta}(\phi) := \int_{I^{\mathbb{Z}^d}} \frac{(\theta_l - \vartheta_l) \cdot \chi_r(\theta - \vartheta)}{|B|(|B|-2)\alpha_{|B|} \cdot \|\theta^B - \vartheta^B\|^{|B|}} \phi(\vartheta) d\vartheta^B \otimes_{j \notin B} \mu_0(d\vartheta_j)$$

and $\mu_{l,r}^{\theta}(f) := \nu_l^{\theta}(\phi) - \nu_{l,r}^{\theta}(\phi)$. A direct computation shows that $|\mu_{l,r}^{\theta}| \leq C r (A C_0 \ln N)^{-d}$ and $\|\nu_{l,r}^{\theta}\| \leq C (A C_0 \ln N)^{-d} \ln r^{-1}$. Since $|\mathbf{f}| \leq 2|g|_{\infty}$ we can finally use Theorem

²⁷Remember that the marginal of $(\mu^e)_{\ell}^{XY}$ on $B \cup B'$ is absolutely continuous with respect to Lebesgue, hence the boundary of I^B has zero measure, moreover $\partial_{\theta_l} \Psi$ is a continuous function on $I^{\mathbb{Z}^d}$.

2.2 to estimate

$$\begin{aligned}
\left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{k-A \ln N}^{XY}) \right| &\leq C \left\{ r + \sup_{\substack{l \in B \\ \theta \in I^B}} |(\tau^{X_k})' \nu_{l,r}^\theta(S^{A \ln N} \tilde{g}_i)| \right\} \\
&\leq C \left\{ r + \mu^w(\tilde{g}_i) + \sup_{\substack{l \in B \\ \theta \in I^B}} \|(\tau^{X_k})' \nu_{l,r}^\theta\| \eta^{A \ln N} \right\} \\
&\leq C \{r + N^{-1} \ln r^{-1}\} \leq CN^{-1} \ln N
\end{aligned}$$

where we have chosen $r = N^{-1}$.

In conclusion,

$$\begin{aligned}
(5.9) \quad & \left| \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i \partial_{x_{iy}} \psi(t, \hat{X}_t^N, \hat{Y}_t^N)(\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j) \right| \leq CN^{-\frac{1}{2}} \|\psi\|_{C^2} \\
& + A \ln N N^{-1} \mathbb{E}^2(\text{Card}\{t \leq Nh : \|X_t - Y_t\| \leq 4C_0 A \ln N\}) \|\psi\|_{C^2}.
\end{aligned}$$

In Section 6 we prove the following bound.

Lemma 5.2. *Let A be a large constant and set $L_N := A \ln N$. There exists $\delta_0 \in (0, 1)$ such that*

$$(5.10) \quad \mathbb{E}^2(\text{Card}\{t \leq N : \|X_t - Y_t\| \leq L_N\}) \leq CN^{\delta_0} \quad (N \in \mathbb{N})$$

Lemma 5.2 allows to estimate the cross term by

$$Ch^{\delta_0} N^{\delta_0-1} \ln N = Ch(hN)^{\delta_0-1} \ln N$$

proving the proposition by choosing $\beta \in (0, \frac{1-\delta_0}{4-2\delta_0})$ and $\vartheta = \frac{2\beta}{1-\delta_0}$. $\square$

6. TWO WALKS ESTIMATES

In Section 5 we proved that Lemma 2.8 (and hence Theorem 2.7) holds provided the average number of times two walks come closer than $A \ln N$ in time N is smaller than N^{δ_0} for some $\delta_0 \in (0, 1)$. The purpose of this section is to prove such an estimate and therefore conclude the argument.

6.1. On the number of close encounters. The proof of inequality (5.10) can be reduced to the following simpler inequality.

Lemma 6.1. *There exists $\rho \in (0, 1)$, $C_7 > 0$ such that for any $m \in \mathbb{N}$ for any a, b such that $\|a - b\| > L_N$, we have*

$$(6.1) \quad \mathbb{P}^2(\{\|X_j - Y_j\| > L_N \quad \forall j \in \{m, \dots, m+N\}\} \mid X_m = a, Y_m = b) \geq \frac{C_7}{N^\rho},$$

(Here $\mathbb{P}^2$ is the underlying probability for the process (θ_t, X_t, Y_t) started with θ_0 distributed according to μ^ϵ).

We postpone the proof of the above Lemma until finishing the proof of (5.10).

Proof of Lemma 5.2. Notice that Assumption 4 implies that the walks can move in different directions with positive probability. In particular, there exists $\gamma > 0$ such that for each $a, b \in \mathbb{Z}^d$, $m \in \mathbb{N}$ and $\delta > 0$,

$$(6.2) \quad \mathbb{P}^2\left(\left\{\|X_{m+L_N^2} - Y_{m+L_N^2}\| \geq L_N\right\} \mid X_m = a, Y_m = b\right) \geq c(\delta)\gamma^{\delta L_N},$$

Indeed

$$\mathbb{P}^2 \left(\left\{ \|X_{m+\delta L_N} - Y_{m+\delta L_N}\| \geq \delta L_N \right\} \mid X_m = a, Y_m = b \right) \geq \gamma^{\delta L_N}$$

the latter being the probability of one fixed path in which X_i, Y_i get further and further apart at each step. On the other hand

$$(6.3) \quad \mathbb{P}^2 \left(\left\{ \|X_{m+L_N^2-\delta L_N} - Y_{m+L_N^2-\delta L_N}\| \geq L_N \right\} \mid \|X_m - Y_m\| \geq \delta L_N \right) \geq c(\delta),$$

To verify (6.3) let $W^{(1)}(t)$ and $W^{(2)}(t)$ be independent Brownian Motions such that $W^{(2)}(0) - W^{(1)}(0) = \mathbf{v}$, where $\|\mathbf{v}\| = \delta$. Let $\hat{\mathbf{v}} := \mathbf{v} \|\mathbf{v}\|^{-1}$, and

$$c(\delta) := \frac{1}{2} P \left(\|W^{(2)}(1) - W^{(1)}(1)\| > 1, \quad \text{and for all } t \in [0, 1] \right)$$

$$\langle W^{(2)}(t), \hat{\mathbf{v}} \rangle > \langle W^{(2)}(0), \hat{\mathbf{v}} \rangle - \frac{\delta}{3} \quad \text{and} \quad \langle W^{(1)}(t), \hat{\mathbf{v}} \rangle < \langle W^{(1)}(0), \hat{\mathbf{v}} \rangle - \frac{\delta}{3}$$

Observe that the invariance principle established in Lemma 4.2, Lemma 3.5 and the fact that local dynamics are independent implies a two particle invariance principle as long as the walkers explore disjoint regions in the phase space. Therefore the probability that two walkers grow L_N apart exploring disjoint regions of the phase space is at least $c(\delta)$ for large N proving (6.3). Choose $\varrho < 1 - \rho$. Then,

$$(6.4) \quad \begin{aligned} & \mathbb{P}^2 \left(\left\{ \sup_{m \leq i \leq m+N^e} \|X_i - Y_i\| \leq L_N \right\} \mid X_m = a, Y_m = b \right) \\ & \leq \prod_{j=1}^{N^e L_N^{-2}} (1 - c(\delta) \gamma^{\delta L_N}) \leq e^{-c(\delta) \gamma^{\delta L_N} L_N^{-2} N^e} \leq e^{-C N^{e/2}}, \end{aligned}$$

provided that δ is sufficiently small.

Next, consider the sets $B_R^- := \{(x, y) : \|x - y\| \leq R\}$, $B_R^+ := \{\|x - y\| > R\}$ and the stopping times, for $k > 0$,

$$\begin{aligned} s_0 &:= \inf \{j \in \mathbb{N} : j > 0, (X_{j-1}, Y_{j-1}) \in B_{L_N}^-, (X_j, Y_j) \in B_{L_N}^+ \}, \\ s_{2k} &:= \inf \{j \in \mathbb{N} : j > s_{2k-2}, (X_{j-1}, Y_{j-1}) \in B_{L_N}^-, (X_j, Y_j) \in B_{L_N}^+ \}, \\ s_1 &:= \inf \{j \in \mathbb{N} : j > s_0, (X_{j-1}, Y_{j-1}) \in B_{L_N}^+, (X_j, Y_j) \in B_{L_N}^- \}, \\ s_{2k+1} &:= \inf \{j \in \mathbb{N} : j > s_{2k-1}, (X_{j-1}, Y_{j-1}) \in B_{L_N}^+, (X_j, Y_j) \in B_{L_N}^- \}. \end{aligned}$$

Clearly, $s_{2k} < s_{2k+1} < s_{2k+2}$ and $s_k > k$. As $X_0 = Y_0$, these stopping times are adapted to the filtration $\mathcal{F}_t^{XY}$. Note that the s_{2k} are upcrossing times hence $\|X_t - Y_t\| \leq L_N$ for all the $t \in \{s_{2k-1}, \dots, s_{2k} - 1\}$. With this notation, (6.4) implies

$$\mathbb{P}^2 \left(\left\{ \sup_{i \leq N} (s_{2i} - s_{2i-1}) > N^e \right\} \right) \leq N \sup_{i \leq N} \mathbb{P}^2 (\{s_{2i} - s_{2i-1} > N^e\}) \leq N e^{-N^{e/2}}.$$

Let us set $J := \inf \{k \in \mathbb{N} : s_k \geq N\}$, clearly $J \leq N$,

$$(6.5) \quad \mathbb{E}^2(\text{Card}\{n < N : \|X_n - Y_n\| \leq L_N\}) \leq N^2 e^{-N^{e/2}} + N^e \mathbb{E}^2(J).$$

It remains to investigate the length of the intervals of time in which the two walks are further apart than L_N . Let $S_n := \{\sup_{k \leq n} (s_{2k+1} - s_{2k}) < N\}$, and

denote by $\mathcal{F}_{s_{2k}}^{XY}$ the σ -algebra associated to the filtration $\mathcal{F}_t^{XY}$ and the stopping time s_{2k} . Then, by (6.1),

$$\begin{aligned} \mathbb{P}^2(\{J > n\}) &\leq \mathbb{E}^2(\mathbb{1}_{S_n}) = \mathbb{E}^2(\mathbb{1}_{S_{n-1}} \mathbb{P}^2(\{s_{2n+1} - s_{2n} < N\} \mid S_{n-1})) \\ &= \mathbb{E}^2(\mathbb{1}_{S_{n-1}} \mathbb{P}^2(\{s_{2n+1} - s_{2n} < N\} \mid \mathcal{F}_{s_{2n}}^{XY})) \\ &\leq \left(1 - \frac{C_0}{N^\rho}\right) \mathbb{E}^2(\mathbb{1}_{S_{n-1}}) \leq \dots \leq \left(1 - \frac{C_0}{N^\rho}\right)^n. \end{aligned}$$

Thus, letting $1 - \varrho > \alpha > \rho$, it follows that

$$\mathbb{P}^2(\{J > N^\alpha\}) \leq C e^{-C_0 N^{\alpha-\rho}}.$$

which means that $\mathbb{E}^2(J) \leq N^\alpha + N \mathbb{P}^2(\{J > N^\alpha\}) \leq C N^\alpha$. In view of (6.5) this proves (5.10) provided we have chosen δ_0 so that $\varrho + \alpha < \delta_0$. $\square$

Our program is thus completed once we prove (6.1). To this end an intermediate result is needed.

Lemma 6.2. *Given $R > 0$, take two points a_R and b_R such that $\|a_R - b_R\| = R$. Consider two walks starting at a_R and b_R respectively with the environment given by the probability distribution $\nu \in \mathcal{B}$ and define the stopping time $\tau_{\delta,R}$ as the first time $n > 0$ such that*

$$\|X_n - Y_n\| \leq \frac{R}{1+\delta} \text{ or } \|X_n - Y_n\| \geq (1+\delta)R.$$

For each $C_8 > 0$, if $\|\nu\| \leq C_8$, then there exists $R_\delta \in \mathbb{R}_+$, $c_1, c_2 > 0$ such that for each $R \geq R_\delta$

$$\mathbb{P}_\nu^2(\{\|X_{\tau_{\delta,R}} - Y_{\tau_{\delta,R}}\| \geq (1+\delta)R\}) \geq \frac{1}{2+\delta} - c_1 e^{-c_2/\delta}$$

Proof. Let T_R be the first time $n > 0$ such that

$$\max(\|X_n - X_m\|, \|Y_n - Y_m\|) \geq \frac{R}{2}.$$

Then, by Section 4 the pair

$$\left(\frac{X_{\min(T_R, tR^2)} - X_m}{R}, \frac{Y_{\min(T_R, tR^2)} - X_m}{R} \right)$$

is asymptotic, when $R \rightarrow \infty$, to a pair of independent Brownian Motions

$$(W^{(1)}(t), W^{(2)}(t)) \in \mathbb{R}^d \times \mathbb{R}^d \quad W^{(1)}(0) = 0, \quad \|W^{(2)}(0)\| = 1$$

stopped at time T when one of them wanders more than $1/2$ from its starting position.²⁸ Let τ be the first time $\|W^{(1)}(t) - W^{(2)}(t)\| = (1+\delta)$ or $\|W^{(1)}(t) - W^{(2)}(t)\| = (1+\delta)^{-1}$. Recall that

$$P(\{\|W^{(1)}(\tau) - W^{(2)}(\tau)\| = (1+\delta)\}) \geq \frac{1}{2+\delta}$$

(the worst case is then $d = 1$, see e.g. [11], Section XI.1). On the other hand

$$(6.6) \quad P(T < \tau) \leq P(\tau > \delta) + P(T < \delta)$$

²⁸More precisely, the fact that each component is approximately Brownian comes from Section 4 while independence is due to the fact that the walkers explore non-intersecting regions in the phase space and can be proven by the same arguments use to estimate (5.9).

and both terms are $O(e^{-c/\delta})$, the first one because

$$P(\tau > (k+1)\delta^2 | \tau > k\delta^2) \leq \gamma < 1$$

and the second one by Hoeffding's inequality (see e.g. [6]). Now²⁹

$$\begin{aligned} & \mathbb{P}(\{\|X_{\tau_{\delta,R}} - Y_{\tau_{\delta,R}}\| = R(1+\delta)\}) \\ & \geq \mathbb{P}(\{\|X_{\tau_{\delta,R}} - Y_{\tau_{\delta,R}}\| = R(1+\delta) \text{ and } \tau_{\delta,R} < T_R\}) \geq \frac{1}{2+\delta} - c_1 e^{-c_2/\delta}. \end{aligned}$$

□

We now use the following comparison criterion (proved in section 6.2).

Lemma 6.3. *Suppose $\xi_1, \xi_2 \dots \xi_n \dots$ is a random process such that $\xi_n = \pm 1$ and for all n*

$$P(\xi_n = 1 | \xi_1 \dots \xi_{n-1}) \geq p.$$

Let $\tilde{\xi}_1, \tilde{\xi}_2 \dots \tilde{\xi}_n \dots$ be iid random variables such that $\tilde{\xi}_n = \pm 1$, and $P(\tilde{\xi}_n = 1) = p$. Let

$$\mathcal{X}_n = \sum_{j=1}^n \xi_j + \mathcal{X}_0 \quad \tilde{\mathcal{X}}_n = \sum_{j=1}^n \tilde{\xi}_j + \tilde{\mathcal{X}}_0.$$

Then for any $\alpha_1 < \alpha < \alpha_2$

$$P(\mathcal{X}_k \text{ reaches } \alpha_2 \text{ before } \alpha_1 | \mathcal{X}_0 = \alpha) \geq P(\tilde{\mathcal{X}}_k \text{ reaches } \alpha_2 \text{ before } \alpha_1 | \tilde{\mathcal{X}}_0 = \alpha)$$

Recall that by Gambler's Ruin Formula for $p \neq 1/2$

$$(6.7) \quad P(\tilde{\mathcal{X}}_k \text{ reaches } \alpha_2 \text{ before } \alpha_1 | \tilde{\mathcal{X}}_0 = \alpha) = \frac{\left(\frac{p}{1-p}\right)^{\alpha_2-\alpha} - \left(\frac{p}{1-p}\right)^{\alpha_2-\alpha_1}}{1 - \left(\frac{p}{1-p}\right)^{\alpha_2-\alpha_1}}$$

Proof of Lemma 6.1. Let $X_m = a$ and $Y_m = b$ with $\|a-b\| \geq L_N$ and $\kappa \in (\frac{1}{2}, 1)$.

Using ellipticity of Assumption 4 for the first δL_N steps we see that with probability greater than $N^{-c\delta}$ our walkers move distance $(1+\delta)L_N$ apart without getting within distance L_N from each other. Let τ_1 be the first time after m when our walkers move distance $(1+\delta)L_N$ apart and let τ_{n+1} be the first time after τ_n when

$$\|X_j - Y_j\| \geq (1+\delta)\|X_{\tau_n} - Y_{\tau_n}\| \text{ or } \|X_j - Y_j\| \leq (1+\delta)^{-1}\|X_{\tau_n} - Y_{\tau_n}\|.$$

Applying Lemma 6.3 to $\mathcal{X}_n = \frac{\ln \|X_{\tau_n} - Y_{\tau_n}\|}{\ln(1+\delta)}$ with $\alpha_1 = \frac{\ln L_N}{\ln(1+\delta)}$, $\alpha = \alpha_1 + 1$, $\alpha_2 = \frac{\kappa \ln N}{\ln(1+\delta)}$ and using Lemma 6.2, taking into account Lemma 3.5, to estimate the probability of moving apart we conclude from (6.7) that for each $\epsilon > 0$, by choosing δ small and N large enough, the probability that the walkers move distance $N^\kappa L_N$ apart without getting within distance L_N from each other is at least $c\delta N^{-\kappa-\epsilon}$.

Hence there is a polynomially small probability of making an excursion of size $N^\kappa L_N$ before returning to a distance L_N . On the other hand once we have such a big excursion Lemma 2.6 implies that it will take more than N steps to come back, indeed

$$\begin{aligned} & \mathbb{P}^2 \left(\left\{ \inf_{\ell+1 \leq j \leq \ell+N} \|X_j - Y_j\| \leq L_N \right\} \mid \|X_\ell - Y_\ell\| \geq N^\kappa L_N \right) \\ & \leq CN e^{-CN^{2\kappa-1}}. \end{aligned}$$

²⁹Here c_1 should be taken a little bit larger than the implied constants in (6.6) to take into account that our process is only approximated by Brownian Motion.

The last two estimates imply Lemma 6.1 □

6.2. Comparison Lemma.

Proof of Lemma 6.3. Let $U_1, U_2 \dots U_n \dots$ be random variables which are independent and uniformly distributed on $[0, 1]$. Define

$$\xi_n^* = -1 \text{ if } U_n < P(\xi_n = -1 | \xi_1 = \xi_1^*, \dots, \xi_{n-1} = \xi_{n-1}^*) \text{ and } \xi_n = 1 \text{ otherwise.}$$

Also let $\tilde{\xi}_n^* = -1$ if $U_n < 1 - p$ and $\tilde{\xi}_n^* = 1$ otherwise. Let

$$\mathcal{X}_n^* = \sum_{j=1}^n \xi_j^*, \quad \tilde{\mathcal{X}}_n^* = \sum_{j=1}^n \tilde{\xi}_j^*.$$

Then $\{\mathcal{X}_n^*\}$ has the same distribution as $\{\mathcal{X}_n\}$, $\{\tilde{\mathcal{X}}_n^*\}$ has the same distribution as $\{\tilde{\mathcal{X}}_n\}$ and $\mathcal{X}_n^* \geq \tilde{\mathcal{X}}_n^*$. □

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DMITRY DOLGOPYAT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, 4417 MATHEMATICS BLDG, COLLEGE PARK, MD 20742, USA

E-mail address: dmitry@math.umd.edu

CARLANGELO LIVERANI, DIPARTIMENTO DI MATEMATICA, II UNIVERSITÀ DI ROMA (TOR VERGATA), VIA DELLA RICERCA SCIENTIFICA, 00133 ROMA, ITALY.

E-mail address: liverani@mat.uniroma2.it