ERGODIC PROPERTIES OF A MODEL RELATED TO DISORDERED QUANTUM ANHARMONIC CRYSTALS

FRANCESCO FIDALEO AND CARLANGELO LIVERANI

ABSTRACT. We introduce a model suggested by disordered anharmonic quantum crystals. We then investigate in detail the ergodic properties exhibited by such a model.

Mathematics Subject Classification: Primary 82C10, 47A35. Secondary 82B44, 46L55.
Key words: Quantum dynamics; Non equilibrium statistical mechanics; Ergodic theory; Disordered systems; Non commutative dynamical systems.

1. INTRODUCTION

Some recent results [7, 11] have emphasized the possibility to use perturbation theory to investigate the ergodic properties of harmonic crystals in presence of a small local non harmonicity. As a result, it has been proven the existence of non-KMS invariant states, stable with respect to a large class of local perturbations [7]. It is then inevitable to address the question if such a method could be applied to a translational invariant anharmonic crystal, at least in the regime when the anharmonicity per site is very small.

The previously mentioned results rested on some weaker version of L^1 asymptotic abelianess for the unperturbed crystal, whereas the above generalization would require some sort of L^1 space-time asymptotic abelianess for the corresponding harmonic crystal. Unfortunately, harmonic crystals seem not to enjoy such a strong ergodic property. Thus, any investigation in such a direction would require non–perturbative techniques. This state of affairs could support the point of view that the KMS are the only stable states with respect to "reasonable" perturbations.

Date: October 1, 2002.

It is a pleasure to thanks the I.H.E.S., where part of the paper was written, for the kind hospitality. We acknowledge the partial support of INDAM and MURST. Finally, we would like to thanks the anonymous referees for forcing us to substantially clarify the context of the paper.

Conversely, in this paper we show that there exist models (see Section 2 for more details) for which the above mentioned program can be carried out. The physical relevance of such models is debatable, so the issue of the existence or not of stable non–KMS invariant states in physical models cannot be considered solved. In fact, we do not claim that these models describe the precise quantitative behaviour of a disordered anharmonic crystal, yet it is possible that such models capture the qualitative behaviour of physically relevant systems. Hence the present work spells a word of caution on the belief that Gibbs (or KMS) states are the only "stable" states and, therefore, the only physically relevant ones ([9]).

The proposed model is inspired by the work done on disordered systems. Such systems enjoy rather strong time mixing, at least limited to the average expectation of some class of observables (e.g. quadratic polynomials of the positions and momenta, such as two point functions). The main point of the model is that it has a non linear dynamics, yet its ergodic properties can be investigated in some detail. As far as we know, this is one of the first extended non-linear systems for which time mixing properties can be investigated at all.

The plan of the paper is as follows.

Section 2 contains a preliminary description of the model and several explanatory comments. Section 3 collects the needed properties on random Hamiltonians. Section 4 defines precisely the algebra (that is the observables) on which the linear model is constructed. Section 5 contains the precise description of the non linear perturbation. In Section 6 we enlarge the algebra of Section 4 in order to properly define a non linear dynamics. In Section 7 we discuss its ergodic properties.

The paper is supplemented by two appendices. Some needed facts about Anderson localization not present in literature, are proven in Appendix A. Appendix B contains a needed estimate related to a sort of L^1 space–time asymptotic abelianess of the linear system.

2. THE MODEL

Let us consider the Hamiltonian

$$H(q,p)=\sum_{i\in\mathbb{Z}^d}\frac{1}{2}(p_i^2+\kappa_i q_i^2)+\sum_{|i-j|=1}V_*(q_i-q_j) \tag{2.1}$$

where the mass of all particles is equal to one, and the Planck constant $\hbar=1$. In the above formula, the nearest neighbour interaction, described by

$$V_*(x) := \frac{1}{2}x^2 + V(x) \tag{2.2}$$

with $x \in \mathbb{R}^d$, is split into the quadratic part, and the non quadratic part $V(x)$ describing the non linear perturbation. All couples (q_i, p_i) are d-dimensional canonical variables describing the state of a particle with the label i whose rest position, corresponding to $q_i = 0$, is precisely the site i of $\mathbb{Z}^d$. Hence, the expression (2.1) represents the Hamilton function of an infinitely extended anharmonic crystal with a nearest neighbour interaction. Actually, the Hamiltonian (2.1) has a purely formal meaning even if the function V is a bounded one. Yet the corresponding Hamiltonian flow can be rigorously defined.

We will consider a *disordered* system associated to the Hamiltonian (2.1). By disordered we mean that all the elastic constants $\{\kappa_i\}_{i\in\mathbb{Z}^d}$ appearing in the Hamiltonian (2.1), are random variables defined on a standard space (X, P). Namely, disregarding for the moment the non quadratic part of (2.1), let us consider the block (infinite) matrix

$$A = \begin{pmatrix} 0 & I \\ \Delta - K & 0 \end{pmatrix} \tag{2.3}$$

where Δ is the d-dimensional discrete Laplacian given by $(\Delta v)_i = \sum_{|i-j|=1} v_j - 2dv_i$, and $K = K(\xi)$ is a random multiplication operator given, for generic $\xi \in X$, by $(K(\xi)v)_i := \kappa_i(\xi)v_i$.

The block matrix $A(\xi)$ acts componentwise on the double sequence of vectors $\{(q_i, p_i)\}_{i\in\mathbb{Z}^d}$, each of which has d component and it is the generator of the linear flow at fixed environment $\{\kappa_i(\xi)\}_{i\in\mathbb{Z}^d}$. We assume that all κ_i are equidistributed independent random variables with law p, the support of p being a bounded closed interval of the positive real line. In such a way, we get for the sample space $X \equiv \prod_{i\in\mathbb{Z}^d} \mathrm{supp}(p)$, and for the law $P \equiv \prod_{i\in\mathbb{Z}^d} p$.[1]

Such a system can be seen as a collection of Hamiltonians systems for each fixed environment (that is for almost all the realizations of the random variables), but can also be seen as a single Hamiltonian system. This latter point of view being more profitable for us. For the reader's convenience, let us briefly describe it in the simplified case in which the system has only finitely many degree of freedom and the non-linear part is absent. Accordingly, we start with a classical disordered system

[1] One could merely suppose that all κ_i are random variables on a common probability space (X, P) on which the lattice translations act ergodically, see e.g. [10], and [13] for a recent result on the two dimensional lattice.

described by the random Hamiltonian

$$H_\xi(q,p) := \sum_{i\in\Lambda} \frac{1}{2}\Big[p_i^2 + \kappa_i(\xi)q_i^2 + \sum_{|i-j|=1} (q_i - q_j)^2\Big]$$

which takes into account the quadratic part of (2.1) and where, for simplicity, the index set Λ is a finite one (i.e. a box in the standard lattice $\mathbb{Z}^d$).

In order to describe the dynamics of the (classical) associated disordered system, we consider the (infinite dimensional) sympletic space $(\mathcal{V},\Omega)$ consisting of an appropriate linear (sub)space $\mathcal{V}$ of measurable functions on X with values in the phase–space $\mathbb{R}^{2d|\Lambda|}$ equipped with a sympletic form Ω of the type

$$\Omega((q,p),(Q,P)) := \int_X \mu(d\xi)\sigma((q(\xi),p(\xi)),(Q(\xi),P(\xi))) . \tag{2.4}$$

Here σ is the usual 2–form on the phase–space $\mathbb{R}^{2d|\Lambda|}$, and μ is any measure on X in the measure–class determined by P.

The equations of motion[2] $(\dot q(\xi),\dot p(\xi)) = A(\xi)(q(\xi),p(\xi))$ are then determined by the Hamiltonian flow T_t^* associated to the functional Hamiltonian

$$\mathcal{H}(q,p) := \int_X \mu(d\xi)H_\xi(q(\xi),p(\xi)) . \tag{2.5}$$

Therefore, the Hamiltonian flow determines uniquely, almost everywhere, the time–evolution at fixed environment. The dynamics of the random model is thus perfectly described, at least in relation to its linear part, by the above infinite dimensional Hamiltonian system.

In order to have a quantum description of our random medium, one must follow the standard procedure of *quantization*, yet there exist different implementations. Our choice is to quantize the sympletic space $(\mathcal{V},\Omega)$ by specifying the measure in Formulae (2.4), (2.5) as the probability P describing the randomness. In such a way, our quantum system is described by a genuine CCR–algebra: the commutation function between (linear combinations of) fields and conjugate momenta is a multiple of the identity operator determined by the sympletic form (2.4), see Section 4 for details.

This approach differs from the alternative one consisting of first quantizing at fixed environment and then studying the collection of the resulting systems. The algebra of observables in this case can be taken to be the "product" (more precisely a direct integral) of the

[2] See (2.3) for the description of the generator A.

CCR–algebras corresponding to each fixed environment. This is a different algebra than the previous one, in fact it has a non trivial centre and it is not a CCR.[3]

To ensure the ergodic properties we are interested in, we choose, for simplicity, a rather small space $\mathcal{V}$: just the smallest one containing the constant functions and invariant with respect to the linear evolution[4] (see Definition 4.1). Clearly, this limits severely the possible observables and it could certainly be improved, yet it cannot be enlarged as to contain observables related to a single environment, as the systems at fixed environment have poor ergodic properties (see Section 7).

In order to put forward a non–linear model for which it is possible to investigate the ergodic properties by a perturbative approach, we must select a class of invariant states. In fact, we consider only quasi-free states (see Section 4). Such quasi–free states are defined on the Weyl operators as

$$\omega(W(v)) = e^{\mathbb{E}(-\frac{1}{2}B(v,v))}\,, \tag{2.6}$$

where B is a random two–point function which uniquely determine the quasi–free state, and $\mathbb{E}$ denotes the average w.r.t. the randomness. Other typical possibilities could be to consider the *quenched* or *annealed* states.[5] For example, the quenched state can be defined in the

[3] To have a more concrete idea of the situation, consider the trivial case in which $d = 1$, Λ consists of only one point and X is a two point space (that is, just a one dimensional harmonic oscillator with two possible, random, values of the spring). In this case the largest possible $\mathcal{V}$ is isomorphic to $\mathbb{R}^4$. For such a choice the CCR in our quantization scheme is naturally represented on the Hilbert space $L^2(\mathbb{R}^4)$. The generated von Neumann algebra is all $\mathcal{B}(L^2(\mathbb{R}^4))$ (the algebra of the bounded operators). The alternative quantization scheme yields the algebra $\mathcal{B}(L^2(\mathbb{R}^2)) \oplus \mathcal{B}(L^2(\mathbb{R}^2))$. The non trivial center consists of operators with two diagonal blocks proportional to the identity.

[4] In the trivial example of footnote 3 this corresponds to all $\mathbb{R}^4$, yet when X it is not discrete the situation is more subtle.

[5] The quenched state ω_q and the annealed one ω_a at inverse temperature β are defined as

$$\omega_q(A) = \mathbb{E}\left(\frac{\mathrm{Tr}(e^{-\beta H}A)}{\mathrm{Tr}(e^{-\beta H})}\right)\ ;\quad \omega_a(A) = \frac{\mathbb{E}(\mathrm{Tr}\,(e^{-\beta H}A))}{\mathbb{E}(\mathrm{Tr}\,(e^{-\beta H}))},$$

when the r.h.s. exist.

environment–by–environment scheme as[6]

$$\omega_q(W(v)) = \mathbb{E}\left(e^{-\frac{1}{2}B(v,v)}\right) . \tag{2.7}$$

Notice that the quenched state is not quasi-free. To compare the two states one must consider them on some common subalgebra, in our case the algebra generated by the constant functions. On such a subalgebra, the states described by (2.6) are the truncated functionals associated to the quenched states given in (2.7), see [4], Section 5.2.3.[7]

In order to treat the non linear perturbation, we assume that the function V appearing in (2.2) is given by the Fourier transform

$$V(x) := \int_{\mathbb{R}^d} e^{i\lambda\cdot x}\nu(d\lambda) \tag{2.8}$$

of a complex bounded Borel measure ν on $\mathbb{R}^d$. The measure ν should satisfy the reality condition $\nu \circ \mathrm{I}_s = \overline{\nu}$ where the *bar* is the complex–conjugation and I_s is the space–inversion.

In addition, to give a meaning to the quantized version of (2.8), we assume that it is *deterministic*. Deterministic means that its quantized version will be given by $P_V := \int_{\mathbb{R}^d} W(\lambda v)\nu(d\lambda)$, where v is an appropriate constant function, see Section 5 for the precise definition.

To simplify matters, we assume that ν is compactly supported into some box $[-L, L]^d$. Further, we also suppose that the Borel measure ν satisfies the bound

$$L^2|\nu|(1) < \frac{1}{K_*} , \tag{2.9}$$

where K_* is the constant appearing in Lemma B.3. Remark that K_* depends only on the unperturbed Hamiltonian H_0.

Loosely stated, our main results can be summarized as follows.
Each invariant state ω of the type (2.6) *is mixing w.r.t. the linear dynamics (Theorem 4.3). The non linear dynamics relative to an infinitely extended perturbation of type* (5.3) *is well defined (Section 5). Finally we show that, for each such ω, there exists a uniquely determined state ω_∞ which is invariant and mixing w.r.t. the non linear dynamics (Theorem 7.3).*

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[6]With an abuse of notation, here we are denoting by $W(v)$ the direct integral $\int_X^{\oplus} P(d\xi)W(v_\xi)$, see [15], corresponding to the element $\{v_\xi\}$ in the field of CCR–algebras.

[7]In addition, their two point functions coincide even at different times, that is $\partial_\lambda\partial_\mu\omega(W(\lambda v)W(\mu T_t w))|_{\lambda=\mu=0} = \partial_\lambda\partial_\mu\omega_q(W(\lambda v)W(\mu T_t w))|_{\lambda=\mu=0}$.

3. GENERAL PROPERTIES OF RANDOM HAMILTONIANS

The results of the present paper rest on some preliminary properties, together with some assumptions on the random Hamiltonian that, for the reader's convenience, are collected in this section.

Let us consider a random Hamiltonian on $\ell^2(\mathbb{Z}^d)$ of the form

$$H(\xi) := -\Delta + K(\xi) \tag{3.1}$$

as that appearing in the infinite block matrix (2.3). It is well known that the following properties hold true.

(a) For almost all $\xi \in X$, the spectrum of $H(\xi)$ is $[0, 4d] + \mathrm{supp}(p)$;

(b) if $d = 1$, or when $d > 1$ in cases of large disorder (that is when $\mathrm{supp}(p) \subset (k_0, +\infty)$ with k_0 is some critical value), almost all $H(\xi)$ have pure point non discrete spectrum;

(c) under the same conditions as those in (b), we have that the spectral measures $\int_X P(d\xi)\langle\delta_m|E_{H(\xi)}(dx)|\delta_n\rangle$ satisfy, for each Borel set Λ of the real line,[8]

$$\int_X P(d\xi)|\langle\delta_m|E_{H(\xi)}(\Lambda)|\delta_n\rangle| \leq Ae^{-B|m-n|} \tag{3.2}$$

which is nothing else than the *Anderson localization*, see e.g. [1, 10].

In addition, we will need the following properties.

(i) There exist Borel functions $\{\varphi_k\}_{k\in\mathbb{Z}^d}$ such that, for each Borel set $\Lambda \subset \mathbb{R}$, we have

$$\int_X P(d\xi)\langle\delta_m|E_{H(\xi)}(\Lambda)|\delta_n\rangle = \int_\Lambda \varphi_{n-m}(x)dx \tag{3.3}$$

where dx is the Lebesgue measure on the real line.

It is well–known that this condition is satisfied for one dimensional models, see [10], Theorem IX.2. Notice that if (3.3) is satisfied, it follows by polarization that all φ_k belong to $L^1(\mathbb{R}, dx)$ with

$$\int_\mathbb{R} |\varphi_k(x)|dx \leq 8\,. \tag{3.4}$$

(ii) All densities φ_k are absolutely continuous w.r.t. the Lebesgue measure with derivatives φ'_k belonging to $L^p(\mathbb{R}, dx)$ for some $1 < p \leq 2$, and satisfying the bound

$$\int_\mathbb{R} |\varphi'_k(x)|^p dx \leq C(|k|^\alpha + 1) \tag{3.5}$$

for positive numbers C, α.

[8] By $\{\delta_n\}_{n\in\mathbb{Z}^d}$ we designate the standard base of $\ell^2(\mathbb{Z}^d)$.

The above regularity, limited to the integrated density of states (i.e. the case (ii) with $k=0$), was proven in [5] for one dimensional models when the density of the probability measure p is infinitely often differentiable. The result for arbitrary k is obtained in Appendix A.

Remark 3.1. *By the previous considerations there exist one dimensional random Hamiltonians $H(\xi)$ on $\ell^2(\mathbb{Z})$ satisfying all properties listed above.*

We conclude with a result which can be considered as an extension of Fubini Theorem to our situation. Its proof is a straightforward approximation argument.

Lemma 3.2. *Under condition (i), we get for each bounded Borel function F on the real line,*

$$\int_X P(d\xi)\langle\delta_m|F(H(\xi))|\delta_n\rangle=\int F(x)\varphi_{m-n}(x)dx\,.$$

4. CCR–ALGEBRA AND QUASI-FREE STATES

We start by considering, for a fixed $\epsilon>0$, the weighted space W_ϵ made of sequences v such that

$$\sum_{m\in\mathbb{Z}^d}e^{\epsilon|m|}|v_m|<+\infty\,.$$

In the following, we will choose $\epsilon>h$, where h is the constant appearing in Lemma B.1. From now on, we keep fix the weighted space $W_\epsilon\equiv L_0$ as above, L is the direct sum of d copies of L_0, that is

$$L\cong L_0\otimes\mathbb{C}^d\,. \tag{4.1}$$

Notice that the random operator $A(\xi)$ given by (2.3), is an equibounded operator on $L_0\oplus L_0$ for all $\xi\in X$. Our starting point is the symplectic phase–space $L^2:=L\oplus L$. If $v\in L^2$, then it can be written as $v=\{(v^k_m)_j\}$ where $k=1,2$ distinguishes the position from the momentum, $j=1,\dots,d$ corresponds to the d-components of the position and the momentum, and finally $m\in\mathbb{Z}^d$ is the index relative to the site. Such a phase–space is naturally equipped with the norm

$$[\![v]\!]:=\sum_{j,k}\sum_{m\in\mathbb{Z}^d}e^{\epsilon|m|}|(v^k_m)_j|<+\infty\,. \tag{4.2}$$

The phase–space is also equipped with the complex–conjugation C_0, and the commutation function

$$\theta_0(u,v)=\left\langle u,\begin{pmatrix}0&iI\\-iI&0\end{pmatrix}v\right\rangle$$

(see [2, 3]). Further, for each $\xi \in X$, let $A(\xi)$ be the random operator (2.3) on the phase–space (L^2, C_0, θ_0). The associated one–parameter group of Bogoliubov automorphisms is given by

$$T_t(\xi)v := e^{tA(\xi)^*}v\,. \tag{4.3}$$

The group $T_t(\xi)$ is easily computed ([6], Formula (1.2)) obtaining

$$T_t(\xi) = \begin{pmatrix} \cos(H(\xi)^{1/2}t) & -H(\xi)^{1/2}\sin(H(\xi)^{1/2}t) \\ H(\xi)^{-1/2}\sin(H(\xi)^{1/2}t) & \cos(H(\xi)^{1/2}t) \end{pmatrix}. \tag{4.4}$$

Definition 4.1. *A deterministic variable* $\mathbf{v} \equiv (v, \tau) \in L^2 \times \mathbb{R}$ *is a (equivalence class of) Borel function on* X *with values in* L^2*, uniquely determined by*

$$\mathbf{v}(\xi) := T_\tau(\xi)v\,.$$

The space $D(L)$ *will be the algebraic linear span of the deterministic variables.*[9]

A deterministic variable $\mathbf{v}$ will be (with an abuse of notation) also an element of $D(L)$ i.e. a linear combination of generators of $D(L)$. Further, if $\mathbf{v} = \sum_i T_{\tau_i}(\xi)v_i$ we write (again with an abuse of notation)

$$\|\mathbf{v}\| := \sum_i \|v_i\|\,.$$

The space $D(L)$ spanned by the deterministic variables is a phase–space in the terminology of [2, 3], if one defines as C the usual complex–conjugation on functions, and the "commutation function" θ as

$$\theta(\mathbf{u}, \mathbf{v}) := \int_X P(d\xi)\left\langle \mathbf{u}(\xi), \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}\mathbf{v}(\xi)\right\rangle. \tag{4.5}$$

If $\mathbf{v} = (v, \tau)$, the map (4.3) again defines a one–parameter group of Bogoliubov automorphisms $\mathbf{T}_t$ on all of $D(L)$ by

$$\mathbf{T}_t\mathbf{v}(\xi) := T_{t+\tau}(\xi)v\,. \tag{4.6}$$

We assume the self dual CCR–algebra $\mathfrak{A}(D(L), C, \theta)$ *canonically associated to the phase–space* $(D(L), C, \theta)$*, together with the time evolution (4.6) and a suitable class of its quasi–free states (Proposition 4.2), as our starting point for the construction of a non linear model.*

The following proposition describes a class of suitable quasi–free states ([2, 3]) on the CCR.

[9] The main reasons for the above choice is that if one considers the space of all square integrable functions on X with value into L^2, then the dynamics given by (4.3) is quasi periodic. Namely, the whole system does not exhibit any natural ergodic property, being the last however present in real models.

Proposition 4.2. *Let F be a bounded Borel function on the real line satisfying, on a neighborhood of the common spectrum of almost all $H(\xi)$,*

(i) $F(x) > 0$,
(ii) $xF(x)^2 \geq \frac{1}{4}$.

Then the two-point function given by

$$B_F(\mathbf{u},\mathbf{v}) = \int_X P(d\xi) \left\langle \mathbf{u}(\xi), \begin{pmatrix} F(H(\xi)) & \frac{i}{2}I \\ -\frac{i}{2}I & H(\xi)F(H(\xi)) \end{pmatrix} \mathbf{v}(\xi) \right\rangle \tag{4.7}$$

leads to a quasi–free state on the self-dual CCR–algebra $\mathfrak{A}(D(L), C, \theta)$ which is invariant for the one-parameter group of Bogoliubov automorphisms (4.3).

Proof. The proof is quite standard (e.g., see [7], Proposition 4.2), taking into account Lemma 3.2, and is left to the reader. □

It is well known that the real elements $D(L_{\mathbb{R}})$ of $D(L)$ generate by exponentiation, the *C^*–algebra $\mathfrak{W}$ of Weyl operators*[10] associated to the GNS representation of a (fixed) quasi–free state ω_F as above, being ω_F given by

$$\omega_F(W(\mathbf{u})) = e^{-\frac{1}{2}B_F(\mathbf{u},\mathbf{u})}\,.$$

The commutation rule (4.5) translates on the Weyl operators as

$$W(\mathbf{u})W(\mathbf{v}) = e^{-\frac{1}{2}\theta(\mathbf{u},\mathbf{v})}W(\mathbf{u}+\mathbf{v})\,.$$

Next, we fix a quasi–free state ω on the self–dual CCR–algebra $\mathfrak{A}(D(L), C, \theta)$ as above, and consider the *quantum dynamical system* (QDS for short) $(\mathfrak{W}, \alpha^0_t, \omega)$ based, respectively, on the C^*–algebra generated by the Weyl operators of the GNS representation of $\mathfrak{A}(D(L), C, \theta)$ relative to ω, the one–parameter group of automorphisms induced on $\mathfrak{W}$ by the Bogoliubov transformations (4.6), and finally the invariant state ω, see [2, 3].[11] As the state ω is kept fixed during our analysis, we

[10] It is unclear if θ is nondegenerate on $D(L)$, that is if $\mathfrak{W}$ is a simple C^*–algebra. This fact does not affect our analysis.

[11] Let $(\mathfrak{A}, \alpha_t, \omega)$ be a triple consisting of a C^*–algebra $\mathfrak{A}$, a one–parameter group of automorphisms of $\mathfrak{A}$, and a state on $\mathfrak{A}$ respectively. Consider the GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ relative to the state ω. By a *quantum dynamical system* we mean that, for $A \in \mathfrak{A}$, all the maps $t \mapsto \alpha_t A$ are continuous in the s–topology ([14], E 5.5) generated by the folium $\mathcal{F}_\omega$. The last condition is equivalent to the fact that the all the maps $t \mapsto \pi_\omega(\alpha_t A)$ are continuous w.r.t. the strong operator topology of $\mathcal{B}(\mathcal{H}_\omega)$. If the state ω is invariant w.r.t. $\{\alpha_t\}$, such a continuity condition assures that the unitary group $\{U_\omega(t)\}$ implementing the dynamics on $\mathcal{H}_\omega$ in Heisenberg picture, is continuous w.r.t. the strong operator topology. Then, by Stone Theorem, the (selfadjoint, possibly unbounded) Hamilton operator H_ω is well–defined on $\mathcal{H}_\omega$.

suppress all the indices relative to ω. Some of the most relevant ergodic properties of the QDS $(\mathfrak{W}, \alpha_t^0, \omega)$ are summarized in the following

Theorem 4.3. *Let $(\mathfrak{W}, \alpha_t^0, \omega)$ be a QDS as above. Then $(\mathfrak{W}, \alpha_t^0, \omega)$ is asymptotically abelian, and mixing as well.*

Proof. We briefly sketch the proof which is quite similar to the corresponding ones in [7]. Let $\mathbf{u} \equiv (u, \tau)$ and $\tilde{\mathbf{u}} \equiv (\tilde{u}, \tilde{\tau})$ be deterministic variables. One computes

$$\| \, [W(\mathbf{u}), W(\mathbf{T}_t\tilde{\mathbf{u}})] \, \| = 2 \sin \left| \frac{1}{2i} \theta(\mathbf{u}, \mathbf{T}_t\tilde{\mathbf{u}}) \right| ,$$

$$\omega(W(\mathbf{u})W(\mathbf{T}_t\tilde{\mathbf{u}})) = \omega(W(\mathbf{u}))\omega(W(\tilde{\mathbf{u}}))e^{-\frac{1}{2}(\theta(\mathbf{u},\mathbf{T}_t\tilde{\mathbf{u}}) + B_F(\mathbf{u},\mathbf{T}_t\tilde{\mathbf{u}}) + B_F(\tilde{\mathbf{u}},\mathbf{T}_{-t}\mathbf{u}))}$$

where θ and B_F are the two–point functions given in (4.5) and (4.7), respectively. By Lemma 3.2, Formulae (3.4) and (4.4), it straightforward to see after an elementary change of variables, that $\theta(\mathbf{u}, \mathbf{T}_t\tilde{\mathbf{u}})$ as well as $B_F(\mathbf{u}, \mathbf{T}_t\tilde{\mathbf{u}})$ are Fourier transforms of L^1–functions. The proof follows by Riemann–Lebesgue Lemma. □

5. THE NON LINEAR DYNAMICS

In order to define a quantum evolution related to the non linear Hamiltonian (2.1), we start by defining, for $i \in \mathbb{Z}^d$ and $k = 1, 2$, a vector $\mathbf{e}_{i,k}$ made of d deterministic variables as

$$\mathbf{e}_{i,k} := (([i,k,1],0), \ldots, ([i,k,d],0)) \tag{5.1}$$

where

$$([i_0, k_0, j_0]_i^k)_j = \delta_{i,i_0}\delta_{j,j_0}\delta_{k,k_0} .$$

We recall that by definition, $(\mathfrak{W}, \alpha_t^0, \omega)$ is covariantly represented on the separable Hilbert space $\mathcal{H}$,[12] where the linear dynamics α_t^0 is generated (in Heisenberg picture) by a strongly continuous one parameter unitary group

$$\alpha_t^0 A = U_t A U_t^{-1} , \tag{5.2}$$

and the invariant state ω is implemented by an invariant vector $\Omega \in \mathcal{H}$. In such a way, the W^*–dynamical system $(\mathfrak{W}'', \alpha_t^0, \omega_\Omega)$ is automatically singled out if the dynamics α_t^0 is defined by the natural extension of (5.2) to all of $\mathfrak{W}''$. The state ω_Ω is the vector state on $\mathfrak{W}''$ given by Ω.

[12]This can be viewed by considering the self–dual algebra $\mathfrak{A}(R^2, C, \theta)$ based on the vector–valued L^2–space $R := L^2(X, P; \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^d)$. In such a way, $\mathfrak{A}(D(L), C, \theta) \subset \mathfrak{A}(R^2, C, \theta)$ and the assertion follows as in Remark 6.3 of [2].

We want to construct the dynamics associated to the perturbation of the free Hamiltonian given by the formal sum

$$P := \sum_{m\in\mathbb{Z}^d} V_m \tag{5.3}$$

where V_m is given by

$$V_m := \sum_{|m-n|=1} \int_{\mathbb{R}^d} \nu(d\lambda) W(\lambda \cdot (\mathbf{e}_{m,1} - \mathbf{e}_{n,1}))\,. \tag{5.4}$$

In Formula (5.4), $x \cdot y$ is the standard Euclidean inner product in $\mathbb{R}^d$, the Borel measure ν is sufficiently "small and regular", and finally the integral is understood in the weak (or equivalently strong) operator topology. More precisely, we restrict ourselves to measure ν satisfying all conditions listed in Section 2, and in particular the bound (2.9) with K the same positive constant as that appearing in Lemma B.3.

To construct the dynamics we use the usual spatial cut–off. Namely, for each bounded region $\Lambda \subset \mathbb{Z}^d$, let us consider

$$P_\Lambda = \sum_{m\in\Lambda} V_m \tag{5.5}$$

where V_m is given by (5.4). Then, P_Λ is a well–defined element of $\mathfrak{W}''$ ([7], pag. 978). The perturbation α_t^Λ of the linear dynamics α_t^0, is defined on all of $\mathfrak{W}''$. It is the unique solution of the integral equation

$$\alpha_t^\Lambda A = \alpha_t^0 A + i \int_0^t ds\, \alpha_s^\Lambda [P_\Lambda,\, \alpha_{t-s}^0 A]\,, \tag{5.6}$$

see [4], Proposition 5.4.1.

Let $\mathbf{v}$ be a deterministic variable, for $i \in \mathbb{Z}^d$ we define the vector–valued function $\gamma_{\mathbf{v}}^i$ as

$$\gamma_{\mathbf{v}}^i(t) := \frac{1}{2} \int_X P(d\xi) \langle \mathbf{e}_{i,2}, \mathbf{T}_t \mathbf{v}\rangle\,. \tag{5.7}$$

Consider for $j = 1,\dots,d$, $k = 1,2$, and $m,n \in \mathbb{Z}^d$, the elementary deterministic variables $\delta_j \cdot \mathbf{e}_{n,k}$ (as usual δ_j is the delta–function in the place j in $\mathbb{C}^d$), defined taking into account (5.1), together with the vector–valued functions $\{\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}\}$ given according to (5.7). Then, it is interesting to note that for $\mathbf{v} = \{(v_l, \tau_l)\}$, $\gamma_{\mathbf{v}}^m$ can be expressed in the following simple form

$$\gamma_{\mathbf{v}}^m(t) = \sum_{j,k,l,n} ({v_l}_n^k)_j \gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}(t+\tau_l) \tag{5.8}$$

where the above series is absolutely convergent.

Let $\beta=(m,n)$ be made of ordered pair of elements of $\mathbb{Z}^d$. We define the set $I_N := \{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d \mid |m|\leq N;\ |m-n|=1\}$. The set I_∞ will be given by $I_\infty := \{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d \mid |m-n|=1\}$.

Again, taking into account (5.1) and (5.7), we set

$$\mathbf{e}_\beta := \mathbf{e}_{m,1}-\mathbf{e}_{n,1}\,,$$

together with the vector–valued function $\gamma^\beta_{\mathbf{v}}(t) := \gamma^m_{\mathbf{v}}(t)-\gamma^n_{\mathbf{v}}(t)$ and

$$g_\beta(\mathbf{v},t,\lambda) = 2\sin(\lambda\cdot\gamma^\beta_{\mathbf{v}}(t))\,. \tag{5.9}$$

The computation of the commutator in (5.6) is the starting point of our analysis. We get

$$[V_m,\,\alpha^0_t W(\mathbf{v})] = -i \sum_{\substack{\beta=(m,n)\\ |n-m|=1}} \int_{\mathbb{R}^d} \nu(d\lambda) W(\lambda\cdot\mathbf{e}_\beta+\mathbf{T}_t\mathbf{v}) g_\beta(\mathbf{v},t,\lambda) \tag{5.10}$$

where the $\gamma^i_{\mathbf{v}}$ are the vector–valued functions defined by (5.7). Next, we define $\underline{t}^n := (t_n,\,\dots,\,t_1)$, $\underline{\lambda}^n := (\lambda_n,\,\dots,\,\lambda_1)$, and $\underline{\beta}^n := (\beta_n,\,\dots,\,\beta_1)$. Taking into account the analogous formulae in Section 5.2 of [7], we let

$$\begin{aligned} &\mathbf{w}_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n) := \sum_{k=1}^n \lambda_k\cdot\mathbf{T}_{-t_k}\mathbf{e}_{\beta_k}+\mathbf{v} \\ &G_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n) := g_{\beta_n}(\mathbf{v},t_n,\lambda_n) \\ &\times\prod_{k=1}^{n-1} g_{\beta_k}\left(\mathbf{T}_{t_{k+1}}\left[\sum_{j=k+1}^n \lambda_j\cdot\mathbf{T}_{-t_j}\mathbf{e}_{\beta_j}+\mathbf{v}\right], t_k-t_{k+1},\lambda_k\right)\,, \end{aligned} \tag{5.11}$$

and we consider the series

$$\begin{aligned} X_t(\mathbf{v}) = W(\mathbf{T}_t\mathbf{v}) &+ \sum_{n=1}^\infty \sum_{\underline{\beta}^n\in I^n_\infty} \int_0^t dt_n \int_{t_n}^t dt_{n-1}\dots\int_{t_2}^t dt_1 \\ &\int_{\mathbb{R}^{nd}} \nu(d\lambda_1)\dots\nu(d\lambda_n) W\left(\mathbf{T}_t\mathbf{w}_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n)\right) G_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n)\,. \end{aligned} \tag{5.12}$$

Theorem 5.1. *The series given in (5.12) is absolutely norm–summable, uniformly for $t\in\mathbb{R}$. Moreover, it can be obtained as the following pointwise norm–limit, uniform in time:*

$$X_t(\mathbf{v}) = \lim_{\Lambda_N\uparrow\mathbb{Z}^d} \alpha^{\Lambda_N}_t W(\mathbf{v}) \tag{5.13}$$

where $\alpha^{\Lambda_N}_t W(\mathbf{v})$ is the unique solution of the integral equation (5.6) associated to the perturbation P_Λ in (5.5).

Proof. We can compute as in [7], the perturbed dynamics on $\mathfrak{W}''$ associated to (5.5), taking into account (5.10). We obtain

$$\alpha_t^{\Lambda_N} W(\mathbf{v}) = W(\mathbf{T}_t\mathbf{v}) + \sum_{n=1}^{\infty}\sum_{\underline{\beta}^n\in I_N}\int_0^t dt_n \int_{t_n}^t dt_{n-1}\dots\int_{t_2}^t dt_1$$
$$\times\int_{\mathbb{R}^{nd}}\nu(d\lambda_1)\dots\nu(d\lambda_n)W\left(\mathbf{T}_t\mathbf{w}_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n)\right)G_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n)$$

where the series is a well–defined converging one by Lemma B.3.[13]

The proof follows again by Lemma B.3 as the r.h.s. of the above series converges in norm to the corresponding one of (5.12). □

The results of the last theorem, and more precisely (5.13), allow us to define the series (5.12) as describing, at least for the Weyl operators, the non linear dynamics associated to the whole perturbation (5.3).

6. THE C^*–ALGEBRA OF PHYSICAL OBSERVABLES

In the previous section we saw that the C^*-algebra $\mathfrak{W}$ generated by the Weyl operators, cannot be invariant under the time evolution (5.12). On the other hand, $\mathfrak{W}''$ is most likely too large for our purposes, and the dynamics on it may very well have poor ergodic properties. We tackle this problem by enlarging $\mathfrak{W}$ enough to accommodate the non linear dynamics, but not as much as to spoil the ergodic properties we are interested in.

We start by introducing the set

$$F_m := \{\mathbf{h}\equiv\{(h_i,t_i)\}\mid h_i:\mathbb{R}^m\to L^2_{\mathbb{R}},\ t_i:\mathbb{R}^m\to\mathbb{R},$$
$$(h_i,t_i)=(0,0)\text{ for all but finitely many } i\in\mathbb{Z}\}.$$

Clearly, $\mathbf{h}(x)=\sum_i T_{t_i(x)}(\xi)h_i(x)\in D(L_{\mathbb{R}})$ is interpreted as a real deterministic variable.

We then define the triples $\mathfrak{m}:=(m,\mathbf{h},\mu)$ where $m\in\mathbb{N}$, $\mathbf{h}\in F_m$ is made of Borel functions, and finally μ is a "suitable" complex measure on $\mathbb{R}^m$. More precisely, given the set

$$M:=\left\{(m,\mathbf{h},\mu)\;\middle|\;\int_{\mathbb{R}^m}e^{\delta[\![\mathbf{h}(x)]\!]}|\mu|(dx)<+\infty\,,\ \delta>0\right\},$$

we can define, for each $\mathfrak{m}=(m,\mathbf{h},\mu)\in M$,

$$W(\mathfrak{m})=\int_{\mathbb{R}^m}W(\mathbf{h}(x))\mu(dx),$$

[13]The finite sums are well defined as all integrands in the r.h.s. of (5.12) are Borel functions, see Footnote 13 of [7].

where the integrals are meant w.r.t the strong (or, equivalently weak) operator topology of $\mathcal{B}(\mathcal{H})$. Next, we consider the linear set of operators

$$\mathcal{M} := \operatorname{span}\{W(\mathfrak{m}) \mid \mathfrak{m} \in M\} .$$

It is easy to check that $\mathcal{M}$ is a $*$–subalgebra of $\mathfrak{W}''$. We define $\mathfrak{M}$ as the C^*-algebra generated by $\mathcal{M}$. Notice that $\mathfrak{W} \subset \mathfrak{M} \subset \mathfrak{W}''$.

Remark 6.1. *The QDS $(\mathfrak{M}, \alpha_t^0, \omega)$ is asymptotically abelian, and mixing as well.*

The last results easily follows by Theorem 4.3 taking into account the definition of $\mathcal{M}$. Let, for $\mathfrak{m} \in M$,

$$\begin{aligned} X_t(\mathfrak{m}) = &\int_{\mathbb{R}^m} W(\mathbf{T}_t\mathbf{h}(x))\mu(dx) \\ +&\sum_{n=1}^{\infty}\sum_{\underline{\beta}^n\in I_\infty^n}\int_0^t dt_n \dots \int_{t_2}^t dt_1 \int_{\mathbb{R}^{m+nd}} \mu(dx)\nu(d\lambda_1)\dots\nu(d\lambda_n) \\ \times &W\left(\mathbf{T}_t\mathbf{w}_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n)\right) G_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n) . \end{aligned} \tag{6.1}$$

Taking into account all the properties relative to the measures $\nu(d\lambda)$ and $\mu(dx)$, Lemma B.3 tells us that the series (6.1) is absolutely norm–summable, uniformly in time as

$$\|X_t(\mathfrak{m})\| \leq \frac{1}{1 - KL^2|\nu|(1)} \int_{\mathbb{R}^m} e^{aL^{-1}[\![\mathbf{h}(x)]\!]}|\mu|(dx) .$$

Theorem 6.2. *The series (6.1) uniquely defines a one–parameter group of automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ of $\mathfrak{M}$. Moreover, such a group of automorphisms can be obtained for $A \in \mathfrak{M}$, by the following pointwise norm–limit*

$$\alpha_t A = \lim_{\Lambda_N \uparrow \mathbb{Z}^d} \alpha_t^{\Lambda_N} A$$

where $\alpha_t^{\Lambda_N} A$ is the unique solution of (5.6).

Proof. We start by showing that, if $\mathfrak{m} \in M$, then all terms of the series (6.1) belong to $\mathcal{M}$. Hence, $\alpha_t^{\Lambda_N}(\mathcal{M}) \subset \mathfrak{M}$ as the series

$$\begin{aligned} \alpha_t^{\Lambda_N} W(\mathfrak{m}) \equiv &\int_{\mathbb{R}^m} W(\mathbf{T}_t\mathbf{h}(x))\mu(dx) \\ +&\sum_{n=1}^{\infty}\sum_{\underline{\beta}^n\in I_{\Lambda_N}^n}\int_0^t dt_n \int_{t_n}^t dt_{n-1} \dots \int_{t_2}^t dt_1 \int_{\mathbb{R}^{m+nd}} \mu(dx)\nu(d\lambda_1)\dots\nu(d\lambda_n) \\ \times &W\left(\mathbf{T}_t\mathbf{w}_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n)\right) G_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n) , \end{aligned}$$

made of elements which generate $\mathcal{M}$ algebraically, converges in norm, so defines an element of $\mathfrak{M}$. Further, by Lemma B.3

$$\left\|X_t(\mathfrak{m}) - \alpha_t^{\Lambda_N} W(\mathfrak{m})\right\| \leq \sum_{n=1}^{\infty} \sum_{\underline{\beta}^n \in I^n_\infty \setminus I^n_{\Lambda_N}} \int_{\Delta_t^n} d^n t$$
$$\times \int_{\mathbb{R}^{m+nd}} |\mu|(dx)|\nu|(d\lambda_1)\dots|\nu|(d\lambda_n)|G_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n)| \longrightarrow 0$$

as Λ_N increases in order to exhaust all of $\mathbb{Z}^d$. Then on $\mathfrak{M}$, $\alpha_t^{\Lambda_N}$ converges pointwise in norm, to a group of automorphisms which, on a single generator $W(\mathfrak{m})$ of $\mathfrak{M}$, coincides with $X_t(\mathfrak{m})$. □

To conclude, we mention without proof the following

Theorem 6.3. *For every real deterministic variable* $\mathbf{v}$ *and each* $t \in \mathbb{R}$*, the non linear evolution* $\alpha_t W(\mathbf{v})$ *is the unique solution of integral equation*

$$\alpha_t W(\mathbf{v}) = W(\mathbf{T}_t\mathbf{v}) + \int_{[0,t]\times\mathbb{R}^d} ds\, \nu(d\lambda) \sum_{\beta \in I_\infty} \alpha_{t-s}(W(\lambda\cdot\mathbf{e}_\beta + \mathbf{T}_s\mathbf{v}))g_\beta(\mathbf{v}, s, \lambda).$$

7. ERGODIC PROPERTIES OF THE NON LINEAR DYNAMICS

The investigation of ergodic properties of the non linear system is possible as we are able to control the Dyson expansion uniformly in time, see Lemma B.3.

We start by noticing that if we define $\Gamma_t A := \alpha^0_{-t}\alpha_t A$, it is immediate to show that, by Theorem 6.2, Formula (6.1) gives the explicit expression for Γ_t on (topological) generators of $\mathfrak{M}$. Accordingly, the *forward Møller morphism* (the backward Møller morphism Γ_- is constructed analogously) $\Gamma_+ A := \lim_{t\to+\infty} \Gamma_t A$ exists as a pointwise norm–limit and is given by

$$\begin{aligned}
&\Gamma_+ W(\mathfrak{m}) = \int_{\mathbb{R}^m} W(\mathbf{h}(x))\mu(dx) \\
&+ \sum_{n=1}^{\infty} \sum_{\underline{\beta}^n \in I^n_\infty} \int_0^{+\infty} dt_n \dots \int_{t_2}^{+\infty} dt_1 \int_{\mathbb{R}^{m+nd}} \mu(dx)\nu(d\lambda_1)\dots\nu(d\lambda_n) \\
&\times W\left(\mathbf{w}_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n)\right) G_n(\mathbf{h}(x), \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n)\,.
\end{aligned} \tag{7.1}$$

In order to obtain on $\mathfrak{M}$ invariant states w.r.t. the perturbed dynamics, we define for the real deterministic variable $\mathbf{v}$,

$$
a^{\pm}_{n,\underline{\beta}^n}(\mathbf{v}) := \begin{cases} \omega(W(\mathbf{v}))\,, & n=0\,,\\ \int_0^{\pm\infty} dt_n \int_{t_n}^{\pm\infty} dt_{n-1}\dots\int_{t_2}^{\pm\infty} dt_1 \int_{\mathbb{R}^{nd}} \nu(d\lambda_1)\dots\nu(d\lambda_n) & \\ \quad\times\omega(W\left(\mathbf{w}_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n)\right))G_n(\mathbf{v},\underline{t}^n,\underline{\lambda}^n,\underline{\beta}^n)\,, & n>0\,. \end{cases}
$$

together with

$$
a^{\pm}_{n,\underline{\beta}^n}(\mathfrak{m}) := \int_{\mathbb{R}^m} a^{\pm}_{n,\underline{\beta}^n}(\mathbf{h}(x))\mu(dx)
$$

for each $\mathfrak{m} = (m, \mathbf{h}, \mu)$.

It is straightforward to check that, for $\mathfrak{m}\in M$, the formula

$$
\omega_{\pm}(W(\mathfrak{m})) := \sum_{n=1}^{+\infty}\sum_{\underline{\beta}^n\in I^n_\infty} a^{\pm}_{n,\underline{\beta}^n}(\mathfrak{m}) \tag{7.2}
$$

defines (*a priori* different) states on $\mathfrak{M}$ which are invariant w.r.t. the non linear dynamics. Such states can be obtained by the pointwise limit

$$
\omega_{\pm}(A) = \lim_{t\to\pm\infty}\omega(\alpha_t A)\,.
$$

Indeed, as the unit ball of $\mathfrak{M}^*$ is $*$–weakly compact, the nets $\{\omega\circ\alpha_t\}_{t\in\mathbb{R}_\pm}$ have cluster points. But such nets converges to the r.h.s. of (7.2) on the total set $\{W(\mathfrak{m})\mid \mathfrak{m}\in M\}$.

Notice that, by definition,

$$
\omega_{+}(A) = \omega(\Gamma_{+}A) \tag{7.3}
$$

for each element A of $\mathfrak{M}$.

The following lemma is needed in the sequel.

Lemma 7.1. *For each $A\in\mathfrak{M}$ we have*

$$
\lim_{t\to+\infty}\omega_{\pm}(\alpha^0_t A) = \omega(A)\,.
$$

Proof. By a standard approximation trick, it is enough to prove the assertion for each $A = W(\mathfrak{m})$. By Lemma B.3 it is enough to check that, for each $n>0$ and $\mathbf{v}$ a real deterministic variable, $\lim_{t\to+\infty} a^{\pm}_{n,\underline{\beta}^n}(\mathbf{T}_t\mathbf{v}) = 0$. The last claim follows as for all the functions g_β given in (5.9), we get $\lim_{t\to+\infty} g_\beta(\mathbf{T}_t\mathbf{v},\tau,\lambda) = 0$ for all fixed τ, λ, taking into account Lemma B.2. □

Now we introduce the *forward basin of attraction* $\mathcal{B}$ of ω w.r.t. the unperturbed dynamics.

$$
\mathcal{B} := \left\{\varphi\in\mathfrak{W}^*\;\middle|\;\lim_{t\to+\infty}\varphi(\alpha^0_t A) = \omega(A)\,,\ A\in\mathfrak{M}\right\}\,.
$$

Lemma 7.1 tells us that $\{\omega_+, \omega_-\} \subset \mathcal{B}$.

Proposition 7.2. *Let $\varphi \in \mathcal{B}$. Then*

$$\lim_{t\to+\infty} \varphi(\alpha_t A) = \omega_+(A)\,.$$

As an immediate consequence $\omega_+ = \omega_-$.

Proof. For $A \in \mathfrak{M}$ we compute by (7.3),

$$\begin{aligned}|\varphi(\alpha_t A) - \omega_+(A)| \equiv&|\varphi(\alpha^0_t\Gamma_t A) - \omega_+(A)|\\ \leq&\|\Gamma_t A - \Gamma_+ A\| + |\varphi(\alpha^0_t\Gamma_+ A) - \omega(\Gamma_+ A)|\,.\end{aligned}$$

The assertion follows as the last quantities go to zero as t increases.

Further, as $\omega_- \in \mathcal{B}$ by Lemma 7.1, and it is also invariant w.r.t. the non linear dynamics by construction, we have by the first assertion,

$$\omega_-(A) = \omega_-(\alpha_t A) = \lim_{t\to+\infty} \omega_-(\alpha_t A) = \omega_+(A)$$

and the proof is complete. □

Setting

$$\omega_\infty := \omega_+ \equiv \omega_-\,,$$

we can conclude by Proposition 7.2 that ω_∞ is the only state in $\mathcal{B}$ which is invariant w.r.t. the non linear dynamics. We remark also that, by a direct inspection of (7.2), the triple $(\mathfrak{M}, \alpha_t, \omega_\infty)$ defines a QDS according to our definition, see Footnote 11.

Finally, we are ready to prove our last announced theorem.

Theorem 7.3. *The QDS $(\mathfrak{M}, \alpha_t, \omega_\infty)$ is asymptotically abelian, and mixing as well.*

Proof. Recall that the QDS $(\mathfrak{M}, \alpha^0_t, \omega)$ is asymptotically abelian, and mixing as well, see Remark 6.1. Next, for $A, B \in \mathfrak{M}$ we get

$$\lim_{t\to+\infty}[A, \alpha_t B] = \lim_{t\to+\infty}[A, \alpha^0_t(\Gamma_t B - \Gamma_+ B)] + \lim_{t\to+\infty}[A, \alpha^0_t(\Gamma_+ B)] = 0$$

by the asymptotic abelianess of $(\mathfrak{M}, \alpha^0_t)$, taking into account that $\Gamma_t B \to \Gamma_+ B$ in norm.

The proof of mixing follows by the analogous property of $(\mathfrak{M}, \alpha^0_t, \omega)$. Namely,

$$\begin{aligned}&\lim_{t\to+\infty} \omega_\infty(A\alpha_t B) = \lim_{t\to+\infty}\lim_{s\to+\infty} \omega(\alpha_s(A)\alpha_{s+t}(B))\\ =&\lim_{t\to+\infty}\lim_{s\to+\infty} \omega(\alpha^0_s(\Gamma_s A)\alpha^0_{s+t}(\Gamma_{s+t}B)) = \lim_{t\to+\infty}\lim_{s\to+\infty} \omega(\Gamma_s A\alpha^0_t(\Gamma_{s+t}B))\\ =&\lim_{t\to+\infty} \omega(\Gamma_+ A\alpha^0_t(\Gamma_+ B)) = \omega(\Gamma_+ A)\omega(\Gamma_+ B) = \omega_\infty(A)\omega_\infty(B)\,.\end{aligned}$$

The theorem is proven as the backward asymptotic abelianess and the backward mixing follow analogously. □

Appendix A

In the present appendix we show that there exist one dimensional models satisfying the inequality in (3.5) for the densities $\{\varphi_k\}$ given in (3.3). We restrict our analysis to the case when the law p is absolutely continuous w.r.t. the Lebesgue measure with infinitely often differentiable density $f \in \mathcal{D}(0,+\infty)$. We refer the reader to [5] for the basic definitions and results about the *supersymmetric replica trick*.

We start by considering, for positive integer k,l, the interval $I_{k,l} \subset \mathbb{Z}$ given by

$$I_{k,l} := [-(\left[\frac{k}{2}\right] + l), \left[\frac{k+1}{2}\right] + l]\,,$$

together with the associated finite volume random Hamiltonian $H_{I_{k,l}}$ on the probability space $(X_{I_{k,l}}, P_{I_{k,l}})$ recovered from (3.1). The main objects of our analysis is, for $\Im(z) > 0$, the Green functions

$$G_{k,l}(z) := \int_{X_{I_{k,l}}} P_{I_{k,l}}(d\xi)\langle \delta_{-\left[\frac{k}{2}\right]} | (H_{I_{k,l}}(\xi) - z)^{-1} | \delta_{\left[\frac{k+1}{2}\right]}\rangle$$

together with their limit $G_k(z)$ as l goes to infinity.

Let h be the Fourier transform of the law $p(dx)$ appearing in the definition of the random Hamiltonian $H(\xi)$, we set $\beta(x,z) := h(x)e^{izx}$. Define

$$(Tf)(r^2) := -2\int_0^{+\infty} J_0(rs)f'(s^2)sds\,; \quad (Jf)(r^2) := \int_0^{+\infty} J_0(rs)f(s^2)sds$$

where J_0 is the Bessel function of order 0, and $B(z)$ is the multiplication operator by $\beta(\,\cdot\,, z)$.

Lemma A.1. *We have*

$$\begin{aligned} G_{k,l}(z) =& 2i(-i)^k \int_0^{+\infty} [(JB(z))^{\left[\frac{k}{2}\right]}(TB(z))^l 1](r^2) \\ &\times [(JB(z))^{\left[\frac{k+1}{2}\right]}(TB(z))^l 1](r^2)\beta(r^2,z)rdr \end{aligned}$$

where

Proof. Setting $\psi^- := \psi$ and $\psi^+ := \overline{\psi}$, we compute (see [5] for the definition of the *transfer operator* $\mathbf{T}(\Phi_1,\Phi_2)$)

$$\begin{aligned} \int \psi_2^{\pm}\mathbf{T}(\Phi_1,\Phi_2)F(\Phi_2^2)d^2\phi_2 d\overline{\psi}_2 d\psi_2 &= \left(\pm i\psi_1^{\pm}\right)\left(\frac{1}{2\pi}\int_{\mathbb{R}^2} f(\phi_2^2)e^{-i\phi_1\cdot\phi_2}d^2\phi_2\right) \\ &\equiv \left(\pm i\psi_1^{\pm}\right)\int_0^{+\infty} J_0(rs)f(s^2)sds\,. \end{aligned}$$

The proof follows as in [5], Section 2. $\square$

As far as the Bessel function of order zero is concerned, we note that

$$\sqrt{r}(Jf)(r^2) = H_0(\sqrt{s}f(s^2))(r) \tag{A.1}$$

where H_0 is the Hankel transform of order zero. Then, a little bit of Harmonic Analysis relatively to Hankel Transforms is needed in the sequel. We denote by M the multiplication operator by $\sqrt{r}$ and by B the multiplication operator by the function β. The norms $[\![\cdot]\!]_i$ denote the Banach norms of the spaces $\mathcal{H}_i$ defined in [5], Section 3.

Lemma A.2. *Let $\beta(r^2) \in L^p([0,+\infty), dr)$, with $p > 2$.*
Then $\|M(JB)^{2j}\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_0)} \leq \|\beta(r^2)\|_p^{2j}$.

Proof. The proof is similar to that of Lemma 3.1 of [5]. Let $f \in \mathcal{H}_1$, then

$$\begin{aligned}
&[\![M(JB)^2 f]\!]_0 \equiv \|\sqrt{r}(JBJBf)(r^2)\|_2 = \|H_0(\sqrt{s}\beta(s^2)(JBf)(s^2))\|_2 \\
=&\|\sqrt{r}\beta(r^2)(JBf)(r^2)\|_2 \leq \|\beta(r^2)\|_p \|\sqrt{r}(JBf)(r^2)\|_{(\frac{1}{2}-\frac{1}{p})^{-1}} \\
=&\|\beta(r^2)\|_p \|H_0(\sqrt{s}\beta(s^2)f(s^2))\|_{(\frac{1}{2}-\frac{1}{p})^{-1}} \leq \|\beta(r^2)\|_p \|\sqrt{s}\beta(s^2)f(s^2)\|_{(\frac{1}{2}+\frac{1}{p})^{-1}} \\
\leq&\|\beta(r^2)\|_p^2 \|\sqrt{s}f(s^2)\|_2 \equiv \|\beta(r^2)\|_p^2 [\![Mf]\!]_0
\end{aligned}$$

where we have used (A.1), Plancherel and Hausdorff–Young Formulae for the Hankel transform, and finally Holder Inequality. The repeated application of the above computation leads to

$$[\![M(JB)^{2j} f]\!]_0 \leq \|\beta(r^2)\|_p^{2j} [\![Mf]\!]_0 \leq \|\beta(r^2)\|_p^{2j} [\![f]\!]_1$$

which is the assertion. □

Remark A.3. *By a simple application of the Lebesgue dominated convergence Theorem, if for $r > 0$, $\beta(r^2) < 1$ and decays faster than the inverse of a power, then $\|\beta(r^2)\|_p < 1$ for all $p > 2$ large enough.*

We are ready to prove the announced regularity result.

Theorem A.4. *Let $H(\xi) = -\Delta + K(\xi)$ be a random Hamiltonian on $\ell^2(\mathbb{Z})$ described by identically distributed random variables $K_{ij}(\xi) = \delta_{ij}\kappa_i(\xi)$ with common law $p(dx) = f(x)dx$.*

If $f \in \mathcal{D}(0,+\infty)$, then for each Borel set $\Lambda \subset \mathbb{R}$ we have

$$\int_X P(d\xi)\langle\delta_m|E_{H(\xi)}(\Lambda)|\delta_n\rangle = \int_\Lambda \varphi_{|n-m|}(x)dx$$

where $\varphi_k \in \mathcal{D}(0,+\infty)$ for each $k \in \mathbb{N}$.

Moreover, for every $p \geq 1$ the densities φ_k satisfy

$$\|\varphi_k'\|_p \leq (\mathrm{supp}(\varphi))^{\frac{1}{p}} C_1(k+1)C_2^k$$

where $C_1 > 0$ and $0 < C_2 < 1$ are positive quantities independent on p and k.

Proof. We consider the case $k = 2j$. The odd case follows analogously.

As it was shown in Section 5 of [5], $\eta(x) := \lim_l (TB(x))^l 1$ exists for each $x \in \mathbb{R}$, as a function in $\bigcap_n \mathcal{H}_n$. Moreover, it is infinitely often differentiable with derivative $\frac{d}{dx}\eta(x)$ belonging again to $\bigcap_n \mathcal{H}_n$. Next, by the above consideration and Lemma A.1, we get

$$G_{2j}(x+i0) = 2(i)^{2j+1}\langle M(JB(x))^j \eta(x), B(x)M(JB(x))^j \eta(x)\rangle \tag{A.2}$$

where

$$\langle u, v\rangle := \int_0^{+\infty} u(r^2)v(r^2)r^{-1}dr\,.$$

Notice that, by polarization, φ_{2j} is given precisely by $G_{2j}(x+i0)$, see [5], pag. 229. As f is infinitely often differentiable with compact support, $\beta(s,x) \equiv e^{isx}h(s)$ falls off more rapidly than the inverse of every power. In such a way, for each $k \in \mathbb{N}$, the multiplication operator $M^k B(x)$ is a bounded one. Hence we can differentiate infinitely often the bilinear form (A.2) obtaining the desired regularity property for all the φ_{2j}. Further, after differenting the form (A.2), we easily obtain

$$\|\varphi'_{2j}\|_\infty \leq C_1(2j+1)C_2^{2j}$$

where C_1 does not depend on j, and C_2, being the $\bar{p}$–norm of $h(r^2)$ ($\bar{p} > 2$), can be made smaller than 1, see Lemma A.2 and Remark A.3. $\square$

Appendix B

In the present appendix we prove some estimations related to L^1–asymptotic abelianess, as well as the basic estimate concerning the control of the dynamics uniformly in time.

We start by considering for $j = 1, \dots, d$, $k = 1, 2$, and $m, n \in \mathbb{Z}^d$, the deterministic variable $\delta_j \cdot \mathbf{e}_{n,k}$ defined taking into account (5.1), together with the vector–valued functions $\{\gamma^m_{\delta_j \cdot \mathbf{e}_{n,k}}\}$ given according to (5.7). We denote by $|\cdot|$ (one of the equivalent) norm on $\mathbb{C}^d$.

Lemma B.1. *The functions* $\{\gamma^m_{\delta_j \cdot \mathbf{e}_{n,k}}\}$ *satisfy*

$$\int_{\mathbb{R}} dt\, \|\gamma^m_{\delta_j \cdot \mathbf{e}_{n,k}}(t)\|_{\mathbb{C}^d} \leq De^{-h|n-m|} \tag{B.1}$$

for uniform constants $D, h \in \mathbb{R}_+$.

Further, for each deterministic variable $\mathbf{v}$ *we get*

$$\int_{\mathbb{R}} dt\, \|\gamma^m_{\mathbf{v}}(t)\|_{\mathbb{C}^d} \leq D\|\mathbf{v}\|e^{-h|m|} \tag{B.2}$$

where D *is the same constant as above and we have assumed* $\epsilon > h$, *for* ϵ *appearing in* (4.2).

Proof. We choose a suitable sequence $\{T_{m,n}\}$ of numbers greater than 1 and split the integral in (B.1) into two parts

$$\int_{\mathbb{R}} |\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}| = \int_{|t|<T_{m,n}} |\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}| + \int_{|t|\geq T_{m,n}} |\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}| \,.$$

Taking into account Lemma 3.2 and (3.2), we obtain for the first addendum

$$\int_{|t|<T_{m,n}} |\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}| \leq c_1 T_{m,n} e^{-B|n-m|}\,.$$

To estimate the second addendum we proceed similarly to Lemma B.3 of [7]. We obtain, after an elementary application of the Holder, Haulsdorff-Young and Poincare inequalities,

$$\int_{|t|\geq T_{m,n}} |\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}| = \int_{|t|\geq T_{m,n}} |\mathcal{F}_k(F_k)(t)| = \int_{|t|\geq T_{m,n}} \left|t^{-1}\mathcal{F}_k(F'_k)(t)\right|$$
$$\leq c_2\left(\int_{|t|\geq T_{m,n}} t^{-p}dt\right)^{\frac{1}{p}} \|\varphi'_{n-m}\|_p \leq c_2\frac{(|n-m|^\alpha+1)}{(T_{m,n})^{\frac{p-1}{p}}}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ stand, respectively, for the real and imaginary part of the Fourier transform, the functions F_k can be computed taking into account (4.4), and finally α is given in (3.5). The proof now follows by choosing $T_{m,n} := e^{c_3|n-m|}$ for suitable $c > 0$.

The proof of (B.2) immediately follows by (B.1), taking into account (5.8). $\square$

Lemma B.2. *For each deterministic variable* $\mathbf{v}$*, the (vector valued) function* $\gamma^m_{\mathbf{v}}$ *is a continuous one vanishing at infinity.*

Moreover,

$$\sup_{t\in\mathbb{R}} \|\gamma^m_{\mathbf{v}}(t)\|_{\mathbb{C}^d} \leq E[\![\mathbf{v}]\!]$$

for a uniform constant $E \in \mathbb{R}_+$.

Proof. We are just shown that the $\gamma^m_{\delta_j\cdot\mathbf{e}_{n,k}}$ are Fourier transform of functions whose L^1–norms are equibounded, see (3.4). Then we have the uniform estimation

$$|\gamma^m_{\mathbf{v}}(t)| \leq E\sum_{j,k,l,n} ({v_l}^k_n)_j \equiv E[\![\mathbf{v}]\!]$$

taking into account (5.8). The second statement is immediate, whereas the first one follows by the Riemann–Lebesgue Lemma, taking into account the Lebesgue dominated convergence Theorem. $\square$

Lemma B.3. *There exist $K_*, a > 0$, independent of the measure ν in (2.8), such that, for each real deterministic variable $\mathbf{v}$, the following estimate holds.*

$$\sum_{\underline{\beta}^n \in I_\infty^n} \int_{\Delta_t^n \times \mathbb{R}^{dn}} |G_n(\mathbf{v}, \underline{t}^n, \underline{\lambda}^n, \underline{\beta}^n)| d^n t |\nu|(d\lambda_1) \dots |\nu|(d\lambda_k) \leq (K_* L^2 |\nu|(1))^n e^{aL^{-1}[\![\mathbf{v}]\!]}.$$

Proof. Remembering definitions (5.8), (5.9) and (5.11) we have

$$|G_n(\mathbf{v}, \underline{t}, \underline{\lambda}, \underline{\beta})| \leq 2^n \prod_{j=1}^{n} \left[\sum_{p=j+1}^{n} |\lambda_j| \, |\lambda_p| \, |\gamma^{\beta_j}_{\mathbf{e}_{\beta_p}}(t_j - t_p)| + \left|\lambda_j \gamma^{\beta_j}_{\mathbf{v}}(t_j)\right| \right]$$
$$= \sum_{q=0}^{n-1} 2^n \sum_{\substack{A \subset \{1,\dots,n-1\} \\ \# A = q}} \sum_{\sigma \in \Xi_A^n} \prod_{j \notin A} \left|\lambda_j \gamma^{\beta_j}_{\mathbf{v}}(t_j)\right| \prod_{j \in A} |\lambda_j| \, |\lambda_{\sigma(j)}| |\gamma^{\beta_j}_{\mathbf{e}_{\beta_{\sigma(j)}}}(t_j - t_{\sigma(j)})|,$$

where $\Xi_A^n := \{\sigma : A \to \{2, \dots, n\} \mid \sigma(k) > k\}$. Next, we for each $\sigma \in \{1, \dots, n\}^A$ with $|A| \leq n$ we use the following $n \times n$ matrices

$$[B(A, \sigma)]_{kl} := \begin{cases} \delta_{kl} & k \notin A \\ \delta_{kl} - \delta_{\sigma(k) l} & k \in A \,, \end{cases}$$

to define $\widetilde{\Xi}_A^n = \{\sigma \in \{1, \dots, n\}^A \mid |\det B(A, \sigma)| = 1\}$. Clearly $\Xi_A \subset \widetilde{\Xi}_A^n$. Moreover a permutation $\pi \in \mathbb{P}_n$ that fixes A acts in a natural way on $\sigma \in \{1, \dots, n\}^A$ and it is easily seen that $\widetilde{\Xi}_A^n$ is globally invariant under this action. Hence, (see [7] for details)

$$\sum_{\underline{\beta}^n \in I_\infty^n} \int_{\Delta_t^n \times \mathbb{R}^{nd}} |G_n| \leq \frac{1}{n!} \sum_{\underline{\beta}^n \in I_\infty^n} \sum_{q=0}^{n-1} 2^n \sum_{\substack{A \subset \{1,\dots,n-1\} \\ \# A = q}} \sum_{\sigma \in \widetilde{\Xi}_A^n} \int_{\mathbb{R}^n \times \mathbb{R}^{nd}} \prod_{j \notin A} \left|\lambda_j \gamma^{\beta_j}_{\mathbf{v}}(t_j)\right| \prod_{j \in A} |\lambda_j| \, |\lambda_{\sigma(j)}| |\gamma^{\beta_j}_{\mathbf{e}_{\beta_{\sigma(j)}}}(t_j - t_{\sigma(j)})| \,.$$

Let us now consider the change of variable $(\lambda'_n, t'_n) = (\lambda_n, t_n)$ if $n \notin A$ and $(\lambda'_n, t'_n) = (\lambda_n, t_n - t_{\sigma(n)})$ if $n \in A$. The Jacobian of such a transformation is exactly $|\det B(A, \sigma)|$ which is precisely 1 by definition. Setting $\Xi^n_q = \{1,\dots,n\}^q$ and remembering Lemma B.1,

$$\begin{aligned}\sum_{\underline{\beta}^n\in I^n_\infty}\int_{\Delta^n_t\times\mathbb{R}^{nd}}|G_n| \leq &\frac{2^n}{n!}\sum_{\underline{\beta}\in I^n_\infty}\sum_{q=0}^{n-1}\binom{n}{q}\sum_{\sigma\in\Xi^n_q}\int_{\mathbb{R}^{nd}}\prod_{j=q+1}^{n}|\lambda_j|\int_{\mathbb{R}}ds\,\|\gamma^{\beta_j}_{\mathbf{v}}(s)\|_{\mathbb{C}^d}\\ &\times\prod_{j=1}^{q}|\lambda_j|\,|\lambda_{\sigma(j)}|\int_{\mathbb{R}}ds\,\|\gamma^{\beta_j}_{\mathbf{e}_{\beta_{\sigma(j)}}}(s)\|_{\mathbb{C}^d}\\ \leq&\frac{2^nD^nd^n}{n!}\sum_{q=0}^{n-1}\binom{n}{q}[\![\mathbf{v}]\!]^{n-q}L^{n+q}e^{(n+q)h}d^{\frac{n+q}{2}}|\nu|(1)^n\\ &\times\sum_{\sigma\in\Xi^n_q}\sum_{\underline{n}\in\mathbb{Z}^n}\prod_{j=1}^{q}e^{-h|n_j-n_{\sigma(j)}|}\prod_{j=q+1}^{n}e^{-h|n_j|}.\end{aligned}$$

To continue we relabel the $\underline{n}$ sum with the new labels $n'_j = n_j$ for $j\leq q$ and $n'_j = n_j - n_{\sigma(j)}$ for $j\leq q$, and let $C_* := \frac{2D}{1-e^{-h}}$.

$$\begin{aligned}\sum_{\underline{\beta}\in I^n_\infty}\int_{\Delta^n_t\times\mathbb{R}^n}|G_n| &\leq \frac{d^nC^n_*|\nu(1)|^n}{n!}\sum_{q=0}^{n-1}\binom{n}{q}[\![\mathbf{v}]\!]^{n-q}L^{n+q}e^{(n+q)h}d^{\frac{n+q}{2}}\sum_{\sigma\in\Xi^n_q}1,\\ &\leq \frac{d^{\frac{3n}{2}}C^n_*|\nu|(1)^nL^ne^{nh}}{n!}\sum_{q=0}^{n-1}\binom{n}{q}[\![\mathbf{v}]\!]^{n-q}L^qn^qe^{qh}d^{\frac{q}{2}} \leq \left(K_*L^2|\nu|(1)\right)^n e^{\frac{a[\![\mathbf{v}]\!]}{L}},\end{aligned}$$

where $K_* := C_*d^2e^{2h+1}$ and $a := \frac{e^{-h}}{\sqrt{d}}$. $\square$

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Francesco Fidaleo, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: fidaleo@mat.uniroma2.it

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: liverani@mat.uniroma2.it