COMPUTING THE RATE OF DECAY OF CORRELATIONS IN EXPANDING AND HYPERBOLIC SYSTEMS

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Abstract. I discuss a general approach allowing to accurately investigative the statistical properties of expanding and hyperbolic dynamical systems.

1. Different approaches

The study of the statistical properties of dynamical systems, and in particular the study of the rate of convergence to equilibrium has a rather long story. In the last years several approaches have been developed, here is a brief (and incomplete) summary

- Coding the system:
  - Markov Partition (this is the original one, see [3] for an overview);
  - Towers (in the same spirit of coding the systems but much more flexible [17]).
- Functional Analysis:
  - Quasi-compactness (this has a rather long story, see [1] and references therein for an overview);
  - Hilbert metrics (see [10] and [16] for an overview).
- Probability:
  - Random approximations (not so systematic as the previous, see [13], [11] and [14] for some examples);
  - Coupling (introduced recently by [4], [18], see also [5]).

Of all the above points of view the functional analytic one turns out to be the most powerful although there is a price to pay: it applies only to systems with uniform expansion and contraction (that is Anosov systems). With this limitation in mind let me proceed in the exposition of such an approach.

2. Functional approach

It turns out to be helpful to expose the approach in a rather abstract fashion. To fix ideas it may be convenient to think of a very simple example *Piecewise expanding maps*, that is a piecewise smooth map $T : [0,1] \to [0,1]$ such that $|DT| \geq \lambda > 1$. Accordingly, I will point out how the general features specialize to such a case.

The ingredients of the functional approach are the following:

1. Two Banach spaces $\mathcal{B}_i$. ($\mathcal{B}_1 = BV$ and $\mathcal{B}_2 = L^1$, in our example).
2. Two norms $\|\cdot\| \geq |\cdot|$. ($\|\cdot\| = \|\cdot\|_{BV}$; $|\cdot| = |\cdot|_1$, in our example).
3. An operator $\mathcal{L} : \mathcal{B}_i \to \mathcal{B}_i$. ($\mathcal{L}$ is the Transfer operator, see below).

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4. An inequality
$$\|\mathcal{L}^n f\| \leq A\alpha^n\|f\| + B|f|, \quad |\mathcal{L}^n f| \leq C|f|$$
with $\alpha < 1$. (Lasota-Yorke inequality, this captures the regularization properties of $\mathcal{L}$).
5. Compactness: $\{f \in \mathcal{B}_1 \mid \|f\| = 1\}$ is pre-compact in $\mathcal{B}_2$. (In our example, this is the compactness of the unit ball of BV in L^1.).
6. Invariant functional: exists $\ell \in \mathcal{B}_2^*$ such that $\mathcal{L}^*\ell = \ell$. ($\ell(f) = \int f(x)dx$, in our example.).

The above setting has some immediate implications:

- (4) $\Longrightarrow \sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \mid |z| \leq 1\}$.
- (6) $\Longrightarrow 1 \in \sigma(\mathcal{L})$.
- (4,5) $\Longrightarrow \sigma_{ess}(\mathcal{L}) \subset \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq \alpha \right\}$

The first two of the above statements concerning the spectrum of $\mathcal{L}$ are obvious, while the last is rather deep and has been obtain, in different contexts, by various authors. In the present form it follows form [6].[1]

In the piecewise expanding case
$$\mathcal{L}f(x) = \sum_{y \in T^{-1}x} |D_y T|^{-1} f(y).$$
Let $T_*\varphi := \varphi \circ T$ and $T^*\mu(\varphi) := \mu(T^*\varphi)$. Setting $d\mu = hdx$
$$\frac{dT^*\mu}{dx} = \mathcal{L}h,$$
that is $\mathcal{L}$ describes the evolution of the densities under the dynamics.

If 1 is a simple eigenvalues and no other eigenvalues of modulus one are present then the projection Π_1 on the associated eigenvector is given by
$$\Pi_1 f = h\ell(f)$$
where $\mathcal{L}h = h$.

Remark 1. *For piecewise expanding maps this corresponds to the uniqueness of the invariant measure and the exponential rate of convergence to equilibrium (given by the spectral gap).*

3. Perturbations

To study more precisely the spectral properties of $\mathcal{L}$, and hence its statistical properties, it would be convenient to approximate $\mathcal{L}$ by a finite rank operator. Unfortunately it is not known how to do this in a way allowing to apply standard perturbation theory. Yet, it is possible to develop a perturbation theory suited to our needs.

Define the norm
$$|||\mathcal{L}||| := \sup_{\|f\| \leq 1} |\mathcal{L}f|.$$
Let $\mathcal{L}_i$ be two operators that satisfy (1-6), and
$$|||\mathcal{L}_1 - \mathcal{L}_2||| \leq \varepsilon.$$

[1] By σ_{ess} I mean the essential spectral radius in the sense that the spectrum outside it consists of a finite number of eigenvalues with finite multiplicity.

In addition, for any operator $\mathcal{L}$, $r > \alpha$ and $\delta > 0$, consider the set

$$V_{\delta,r}(\mathcal{L}) := \{z \in \mathbb{C} \mid |z| \leq r \text{ or } \operatorname{dist}(z, \sigma(\mathcal{L})) \leq \delta\}.$$

Since the complement of $V_{\delta,r}(\mathcal{L})$ belongs to the resolvent of $\mathcal{L}$ it follows that

$$H_{\delta,r}(\mathcal{L}) := \sup\left\{\|(z-\mathcal{L})^{-1}\| \mid z \in \mathbb{C}\backslash V_{\delta,r}(\mathcal{L})\right\} < \infty.$$

Theorem 1 ([8]). *Consider two operators $\mathcal{L}_i : \mathcal{B}_1 \to \mathcal{B}_1$ satisfying (1-6). Let $H^i_{\delta,r} := H_{\delta,r}(\mathcal{L}_i)$; $V^i_{\delta,r} := V_{\delta,r}(\mathcal{L}_i)$, then, if $\varepsilon \leq \varepsilon_1(\mathcal{L}_1, r, \delta)$,*

$$\|(z-\mathcal{L}_2)^{-1} f\| \leq \frac{4(A+B)}{1-r} r^{-n_1}\|f\| + \frac{1}{2\varepsilon_1}|f|.$$

In addition, setting $\eta := \frac{\ln r/\alpha}{\ln \alpha^{-1}}$ there exists $a, b \in \mathbb{R}^+$ such that, if $\varepsilon \leq \varepsilon_0(\mathcal{L}_1, r, \delta)$, for each $z \in \mathbb{C}\backslash V^1_{\delta,r}$ holds

$$|||(z-\mathcal{L}_1)^{-1} - (z-\mathcal{L}_2)^{-1}||| \leq \varepsilon^\eta \left[a\|(z-\mathcal{L}_1)^{-1}\| + b\|(z-\mathcal{L}_1)^{-1}\|^2\right].$$

In our example the previous theorem applies to operators $\mathcal{L}_\varepsilon$, $\mathcal{L}$ such that

1. The $\mathcal{L}_\varepsilon$ are Perron-Frobenius operators of maps T_ε which are "close" to T;
2. $\mathcal{L}_\varepsilon$ is the transition operator of the stochastically perturbed map T, where ε is the "size" of the perturbation;
3. $\mathcal{L}_\varepsilon$ is the transition operator for the Ulam-type discretization of T with grid size ε;
4. $\mathcal{L}_\varepsilon$ is the transfer operator for the map with a small hole;

and implies the **stability of the peripheral spectrum** and of the associated eigenprojections.

In particular, point (3) allows to *compute* the peripheral spectral properties of $\mathcal{L}$. Let $\{\phi_i\}$ be a partition of unity with small support ($\leq \varepsilon$) then

$$\mathcal{L}_\varepsilon f(x) := \sum_{ij} \phi_i(x) Z_i Z_j \int \phi_i \mathcal{L}\phi_j \int \phi_j f; \quad Z_i^{-1} = \int \phi_i.$$

is a finite rank operator which spectrum and eigenvectors are close to the ones of $\mathcal{L}$, in the $|\cdot|$ norm. So one can get precise quantitative informations on (see [12] for more details)

- The ergodic properties (multiplicity of the eigenvalue 1, eigenvalues of modulus one...);
- The invariant measure (knowledge of the density in L^1 norm);
- The rate of decay of correlations (spectral gap).

4. Anosov maps

The above abstract setting has been illustrated in the case of piecewise expanding maps of the interval, but it has a much wider scope. To have an idea of the possibilities let us apply it to the case of Anosov systems (see [7] for definitions and properties of such systems). The following results are taken by [2].

Let $T : \mathcal{M} \to \mathcal{M}$ be C^2 and Anosov, then

$$\mathcal{L}f = |\det(D_x T^{-1})| f \circ T^{-1} \quad \forall f \in L^1(\mathcal{M}, m)$$

For the Banach space on which the operator acts must contain the density of the invariant SRB measure and since such a measure is, in general, not absolutely continuous with respect to Lebesgue, it follows that we need a space of distributions. To define it we need first suitable test functions.

For $\varphi : \mathcal{M} \to \mathbb{R}^d$, measurable, define[2]

$$
\begin{aligned}
H^s_\beta(\varphi) &:= \sup_{d^s(x,y)\le\delta} \tfrac{|\varphi(x)-\varphi(y)|}{d^s(x,y)^\beta},\\
|\varphi|_\beta &:= |\varphi|_\infty + H^s_\beta(\varphi).
\end{aligned}
$$

Let $\mathcal{V}$ be the set of measurable vector fields on $\mathcal{M}$ such that $v(x) \in E^u(x)$.

For $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{R})$ let

$$
\begin{aligned}
\|f\|_s &= \sup_{|\varphi|_\beta \le 1} \int_{\mathcal{M}} f\varphi\, dm,\\
\|f\|_u &= \sup_{v\in\mathcal{V};\ |v|_\beta\le 1} \int_{\mathcal{M}} d_x f(v(x)) m(dx),\\
\|f\| &= \|f\|_u + b\|f\|_s,\\
|f| &= \sup_{|\varphi|_1\le 1} \int_{\mathcal{M}} f\varphi\, dm.
\end{aligned}
$$

Then $\mathcal{B}_i \subset \mathcal{C}^1(\mathcal{M},\mathbb{R})^*$ are the completion of $\mathcal{C}^1(\mathcal{M},\mathbb{R})$ with respect to the above norms.[3]

It is possible to show that there exists $\sigma \in (0,1)$, $B > 0$, such that

$$
\begin{aligned}
|\mathcal{L}f| &\le |f|,\\
\|\mathcal{L}f\| &\le \sigma\|f\| + B|f|_w.
\end{aligned}
$$

The unit ball of $\mathcal{B}_1$ is pre-compact in $\mathcal{B}_2$. $1 \in \mathcal{B}_2^*$ and $\mathcal{L}^*1 = 1$.

That is points 1-6 of the abstract setting apply. All the usual results about SRB measures easily follow.

To perform the perturbative part of the argument we need more stringent conditions on the systems: it is necessary to require that the invariant foliations be $C^{1+\alpha}$. This is automatically verified for two dimensional systems, so I will restrict the following discussion to such simple case. Set $\mathcal{M} = \mathbb{T}^2$. Consider $\bar{q} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, supp $\bar{q} \le B_1(0)$, $\int \bar{q} = 1$. Define the convolution

$$
Q_\varepsilon f(x) = \int_{\mathbb{T}^2} \varepsilon^{-2}\bar{q}(\varepsilon^{-1}(x-y))f(y)m(dy).
$$

Combined with the deterministic transformation T such operator gives rise to a Markov process $(X_n^\varepsilon)_{n\ge 0}$ on $\mathbb{T}^2$ where X_0 can have any probability distribution and

$$
P(X^\varepsilon_{n+1} \in A \mid X^\varepsilon_n) = \int_A \varepsilon^{-2}\bar{q}(\varepsilon^{-1}(T(X_n)-y))\, m(dy)\ .
$$

The associated transfer operator is

$$
\mathcal{L}_\varepsilon = Q_\varepsilon \mathcal{L}.
$$

It is then possible to show that there exists a constant $K > 0$ such that, for ε small enough,

$$
\begin{aligned}
&|\mathcal{L}^n_\varepsilon f| \le K|f| \quad \forall n \in \mathbb{N},\\
&\|\mathcal{L}_\varepsilon f\| \le \sigma^n\|f\| + K|f| \quad \forall n \in \mathbb{N},\\
&|||\mathcal{L}_\varepsilon - \mathcal{L}||| \le K\varepsilon^{1-\beta}.
\end{aligned}
$$

That is the operators $\mathcal{L}$, $\mathcal{L}_\varepsilon$ satisfy all the hypotheses of Theorem 1.

[2] By d^s I mean the distance of two points computed along the stable manifold. In particular, if the two points do not belong to the same stable manifold, then their d^s distance is infinite.

[3] Such functional spaces are vaguely related to the works [15], [9].

As a consequence the following results can be obtained

- Spectral stability with respect to random perturbations;
- Spectral stability with respect to $\mathcal{C}^3$ deterministic perturbations;
- Spectral stability with respect to *Smooth Ulam discretizations*.

The last point is the one we are mostly interested in here. It means that if $\{A_i\}$ is a partition of $\mathbb{T}^2$ with squares of side smaller than ε and $\Pi f := \sum_i \mathrm{Id}_{A_i} Z_i \int_{A_i} f$, with $Z_i^{-1} := \int_{A_i} 1$ (Id_A is the characteristic function of the set A) then the spectrum of the operator $\Pi \mathcal{L} Q_\delta \Pi$ is close with multiplicity to the spectrum of the operator $\mathcal{L}$ provided δ and $\delta^{-5}\varepsilon$ are small. Clearly this allows to reduce the study of the spectrum of $\mathcal{L}$ to the computation of the spectrum of a finite matrix whereby allowing to obtain precise quantitative informations on the ergodic properties of the system under consideration.

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