RETURN TO EQUILIBRIUM IN CLASSICAL AND QUANTUM SYSTEMS

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ABSTRACT. I discuss return to equilibrium both in the classical and quantum setting. In the classical setting the emphasis is on several techniques that have been developed in the last years and that, hopefully, could inspire new approaches in the study of quantum systems. In the quantum setting a very simple, but far from trivial, model is discussed.

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1. First Lecture

The convergence to equilibrium is an ubiquitous phenomena that surrounds us, yet it is hardly well understood.

We have reached a good understanding in the classical setting (although limited to a special class of systems whose physical relevance may be debated) but the quantum setting is very much unexplored. The primary reason of such a lack of understanding in the quantum situation is due to the fact that quantum systems with finitely many degree of freedom typically do not exhibit relaxation to equilibrium (unlikely their classical counterpart). This is readily see by noticing that, if the classical state space is compact (at any given energy), the associated quantum Hamiltonian is normally an operator with only point spectrum. Accordingly the motion is quasi–periodic, hence no asymptotic state can exist. Indeed, as we will see later in more details quasi–periodic motion is associated to (peripheral) point spectrum while convergence to equilibrium needs continuous spectrum.

The above considerations imply that, in order to have convergence to equilibrium in quantum systems, it is necessary to have systems with 'non compact' classical phase space (thus either on a non compact region or with infinitely many degree of freedom). This is quite reasonable from a physical point of view,[1] yet rather problematic from the mathematical point of view. In fact, even classically the theory of such systems is quite unsatisfactory.

The aim of these notes is to present some of the results of the classical theory in a form that is reminiscent of the quantum setting in the hope that this could shed some light on the latter. We will also review some results on classical infinite systems and mention few results in the quantum mechanical framework. As the reader will see the lesson learned in the classical situation has no immediate bearing on the quantum one. In fact, the main goal of these lectures is to point out this state of affairs and pose the problem to find out if some of the ideas (reviewed here) that have proved fruitful for classical systems could be applicable (in some properly revised form) to quantum systems.

1.1. Classical Systems: Introduction.

The above mentioned class of classical systems for which it is possible to investigate the convergence to equilibrium are the so called *hyperbolic systems*. Generally speaking this are systems that exhibit a strong sensitivity with respect to the initial conditions, so that there is some 'memory loss'.[2]

The traditional approach to studying a classical system is to investigate directly the motion in the phase space. This is a first drastic difference from quantum mechanics where the phase space does not exists. In fact, the phase space enters in the very definition of a classical dynamical system.

Definition 1. *A classical discrete (continuous) measurable dynamical system consists of a measurable space X, a measurable map T (a measurable flow ϕ^t) and an invariant measure μ.*

[1] Think for example of radiation: an atom is excited by some external factor, after that goes back to its fundamental state by emitting radiation. Clearly, the electromagnetic fields is exactly the system with infinitely many degree of freedom we are advocating.

[2] Of course, since these systems are deterministic there cannot be any real 'memory loss'. What we mean here is that if the initial condition is known only with a given precision (which is always the case if the observable is a smooth function), then after some time we are no longer able to predict the state of the system due to the magnification of the original uncertainty.

Nevertheless, more abstract approaches, based directly on the evolution of the observables or of the measures (which are available in quantum mechanics as well), have been developed recently.

A seminal example of the above point of view is the Mean Ergodic Theorem.[3] In this theorem one considers the space $L^2(X, \mu)$ as the space of observables[4] and defines on it the action of the dynamics via the natural formula

$$T_* f := f \circ T \quad (\phi^t_* f := f \circ \phi^t \text{ in the continuous case}) \tag{1.1}$$

It is easy to see that the evolution operator (due to the invariance of the measure) is a contraction (in fact unitary if T is invertible or if ϕ_t is a group).

This point of view has been furthered considerably by the so called *Russian School*[5] that has clarified the relation between the spectral properties of the above operators and the ergodic properties of the underlying systems. Here it suffices to say that the spectrum is contained in the unit circle and that the statistical properties of the system are related to the structure of the peripheral spectrum (that is $\sigma(T_*) \cap \{z \in \mathbb{C} \mid |z| = 1\}$). Namely, if the eigenspace corresponding to one is one dimensional, then the map is ergodic.[6] In addition, the system is mixing iff one is the only element of the point spectrum of modulus one.[7] Many more informations can be extracted by studying the spectrum of T_* on L^2 (e.g., the K-property), see [20] for details.

The limit of the above approach is that the informations obtained are only qualitative, yet it may be not enough to know that a system is mixing: for practical purposes often it is necessary to know something about the speed of mixing.[8] Another related limit is in the choice of the space of observables. A little though shows that there are many limitations on possible observation of a system and this stands in sharp contrast with the above assumption than any function in L^2 represents a valid measurement.[9]

Unfortunately, in general it is not know how to properly restrict the space of observables; yet a first, very fruitful, possibility is given by requiring some smoothness. To have a concrete idea of what can be done let us consider a concrete very simple (but non trivial) case.

1.2. **A model problem.** As already mentioned one can consider systems in discrete or continuous time.[10] In the classical case the relevance of the discrete time

[3]Not by chance due to Von Neuman whose interest in quantum mechanics need not be remarked.

[4]This space is very convenient but has its drawbacks, for example, contrary to the situation in quantum mechanics it is not an algebra.

[5]Composed by such people as Kolmogorov, Sinai, Arnold, Katok etc.

[6]Note that, since $T_*1 = 1$, one is always in $\sigma(T_*)$.

[7]Note that is the map is invertible, hence T_* unitary, the spectrum is contained in $\{z \in \mathbb{C} \mid |z| = 1\}$. As already remarked, in this context mixing is linked to continuous spectrum.

[8]Think of a system with an extremely long transient behavior. Even if such a systems possesses an asymptotic state it could very well happen that all reasonable physical measurements show only the transient, accordingly the asymptotic state would be physically irrelevant.

[9]This problem is particularly acute for infinite systems (thermodynamics) where, for example, the position of atom one thousand billions two hundred three is a particularly unreasonable observable that, nevertheless, belong to the proper L^2.

[10]In fact, from a more abstract point of view one can consider the dynamics as the action of a group (or a semi-group) on the space of observables. Clearly $\mathbb{N}$ or $\mathbb{Z}$ corresponds to discrete time while $\mathbb{R}^+$ or $\mathbb{R}$ to continuous time. In this setting one should mentions that also higher rank Abelian groups make sense, for example in statistical mechanics it is very natural to consider

it is not just due to its simpler structure but also to the possibility to reduce continuous systems to discrete ones via the so called *Poincarè sections*.[11] We will then consider the simplest case of dynamical system with discrete time that exhibit sensitive dependence from initial conditions: an expanding map of the circle.

Let S^1 be the unit circle[12] and $T : S^1 \to S^1$ a $C^2(S^1)$ map with the property there exists $\lambda \in \mathbb{R}^+$ such that

$$\inf_{x \in S^1} |D_x T| > \lambda > 1. \tag{1.2}$$

In addition, we use the Lebesgue measure m as reference measure. This means that we would like to consider only invariant measures that are absolutely continuous with respect to Lebesgue.[13]

The original way to approach the study of such a system was via the introduction of the so called *Markov Partitions*. A Markov partition is nothing else than a partition of the phase space with some extra property that make it 'agree' with the dynamics.[14] The idea is then to use the partition to code the system.[15] After that, statistical mechanics methods are employed to study the resulting symbolic dynamics.

Clearly this approach is not very appealing from our point of view since in makes a very direct use of the phase space and of its geometrical properties. I will not discuss it any further although it is certainly an extremely powerful tool. On the contrary I will present other three approaches that have proven very powerful and, all considered, more general than the one above.

Before discussing this approaches one by one, we need a bit of notation. I have already complained that $L^2(S^1, m)$ is a space too large for our purposes, here I will briefly discuss a possibility that is still unsatisfactory but relevant for the sequel: $C^{(0)}(S^1)$.

The space of continuous function is attracting since its dual consists exactly of the measures and if we have only a statistical knowledge of the system (which often happens in real situations where only a partial knowledge is available) then it will evolve according to the way measures evolve. Clearly the operator T_* (that we defined on $L^2(S^1, m)$) is well defined on the continuous function as well, it is then obvious that the Borel measures $\mathcal{M}(S^1)$ evolve according to

$$T^*\mu(f) = \mu(T_* f) \quad \forall f \in C^{(0)}(S^1). \tag{1.3}$$

To consider the evolution of the measures is a very productive point of view, nevertheless we have already argued that we are not interested in all the measures.

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only measures that are invariant under space translations, hence one can consider space time translations as a unique object, whereby obtaining the action of, e.g., $\mathbb{R}^4$.

[11] A Poincaré section is defined by taking a smooth manifold transversal to the flow and considering the return to it. More precisely, if $\mathcal{M}$ is the manifold (the section), then $T : \mathcal{M} \to \mathcal{M}$ (the Poincarè map) is given by $Tx := \phi_{t(x)}(x)$ where $\phi - t$ is the flow and $t(x)$ is the first time for which $\phi_t(x) \in \mathcal{M}$.

[12] That is $\mathbb{R} \mod 1$.

[13] For example, consider the trivial case $Tx = 2x \mod 1$, then one can easily check that the Lebesgue measure itself is invariant, yet other invariant measures do exists. The most obvious being the delta mass at zero. The reason why we are discarding such a measure is a physical, not a mathematical one.

[14] In the case at hand a Markov partition would be any partition in intervals $\{I_i\}$ such that, for each i, j, $TI_i \cap I_j$ is either empty or equal to I_j. See [13] for more details.

[15] This means that each point $x \in S^1$ is associated to its history $\{T^n x\}_{n \in \mathbb{N}}$ and, more precisely, to the sequence $\sigma \in \{1, ..., m\}^{\mathbb{N}}$, where m is the cardinality of the partition and $\sigma_n = i$ iff $T^n x \in I_i$.

The key to restrict the class of measure we are interested in is to notice that, if μ is absolutely continuous with respect to m, so it is $T^*\mu$. It is then natural to define the operator $\mathcal{L} : L^1(S^1, m) \to L^1(S^1, m)$ by

$$\mathcal{L}h = \frac{dT_*\mu}{dm} \tag{1.4}$$

where $d\mu = hdm$. The operator $\mathcal{L}$ is called Perron-Frobenius or Ruelle-Perron-Frobenius or Transfer operator and it is the one that shows how the density of a measure that is absolutely continuous with respect to Lebesgue evolves. A direct computation immediately shows that

$$\mathcal{L}h(x) = \sum_{y \in T^{-1}x} |D_y T|^{-1} h(y). \tag{1.5}$$

1.3. **A functional approach (quasi-compactness).** The idea here (and in the following) is to study the evolution of the transfer operator, in this particular approach the evolution is studied via the spectral properties of the operator.

Once again we are confronted with the choice of a space of functions, since the spectral properties of an operator depend in a fundamental way on the space on which it is considered.

A lot of work has been devoted to such an issue, here we make the simplest relevant choice: the Sobolev space $W_{1,1}$.[16]

A first relevant fact is that the Transfer operator is an L^1 contraction, indeed

$$|\mathcal{L}h|_1 = \int_{S^1} dx |\mathcal{L}h(x)| \leq \int_{S^1} dx (\mathcal{L}|h|)(x) = \int_{S^1} |h| = |h|_1, \tag{1.6}$$

where we have used the fact that, by definition, if $d\mu = hdm$, then $m(1\mathcal{L}h) = T^*\mu(1) = \mu(1 \circ T) = \mu(1) = m(h)$.

The fundamental fact in the study of the hyperbolic systems is that the Transfer operator has some regularization property. To see it let us do the following easy computation

$$(\mathcal{L}h)'(x) = -\sum_{y \in T^{-1}x} |D_y T|^{-2} D_y^2 T h(y) + \sum_{y \in T^{-1}x} |D_y T|^{-2} h'(y).$$

Thus

$$|(\mathcal{L}h)'|_1 \leq \lambda^{-1} |\mathcal{L}|h'|\,|_1 + B|\mathcal{L}|h|\,|_1 = \lambda^{-1}|h'|_1 + B_0|h|_1, \tag{1.7}$$

where

$$B_0 := \sup_{x \in S^1} \frac{|D_x^2 T|}{|D_x T|}. \tag{1.8}$$

Putting together the previous inequality yields the fundamental Lasota-York inequality

$$\|\mathcal{L}h\| \leq \lambda^{-1}\|h\| + (B_0 + 1)|h|_1. \tag{1.9}$$

From equation (1.9), by iteration, immediately follows

$$\|\mathcal{L}^n h\| \leq \lambda^{-n}\|h\| + \frac{B_0 + 1}{1 - \lambda^{-1}}|h|_1 := \lambda^{-n}\|h\| + B|h|_1 \tag{1.10}$$

[16] Recall that $W_{1,1}$ is nothing else than the completion of $C^{(1)}$ with respect to the Banach norm $\|f\| := \int |f| + \int |f'|$. A further interesting fact, from our point of view, is that such a space is an algebra.

that is, the spectral radius of $\mathcal{L}$ is less or equal than one. On the other hand, as already noted, $m(\mathcal{L}^n h) = m(h)$ thus the spectral radius must be exactly one.

The next step is to notice that the unit ball $B_1 = \{h \in W_{1,1} \mid \|h\| \leq 1\}$ is a compact set in $L^1(S^1, m)$. Thanks to this Lasota and Yorke [58] proved the existence of an invariant measure since the average

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}^i h$$

is a compact sequence and hence it contains converging subsequences in $L^1(S^1, m)$ which define the densities of invariant measures absolutely continuous with respect to Lebesgue.

But much more can be extracted from (1.9), [44]. Let us see this in the simplest possible manner.

Let $\{\varphi_i\} \subset C^{(\infty)}(S^1)$ be a partition of unity such that, calling $I_i := \operatorname{supp}\varphi_i$, all the I_i are contained in an interval of size ε and $I_i \cap I_j = \emptyset$ if $|i - j| > 1$.[17] Define

$$\Pi_\varepsilon f := \sum_i \varphi_i \frac{\int \varphi_i f}{\int \varphi_i} \tag{1.11}$$

Then

$$\begin{aligned}|\Pi_\varepsilon f - f|_1 &\leq \sum_i \int \varphi_i(x) \left| f(x) - \frac{\int f\varphi_i}{\int \varphi_i}\right|_1 \leq \sum_i \int \varphi_i \frac{\int |f(x) - f(\xi)|\varphi_i(\xi)}{\int \varphi_i} \\ &\leq \sum_i \int \varphi_i \int_{I_i} |f'| \leq \varepsilon |f'|_1 \leq \varepsilon \|f\|\end{aligned}$$

The key fact about Π_ε is that it is a compact operator (in fact a finite rank operator) and one of Nussbaum [64] formulae for the essential spectral radius is

$$\sigma_{\text{ess}}(\mathcal{L}) = \lim_{n\to\infty} \frac{1}{n} \ln \inf_{K \text{ compact}} \|\mathcal{L}^n - K\|.$$

But

$$\|(\mathcal{L}^n - \mathcal{L}^n \Pi_\varepsilon)h\| \leq \lambda^{-n}\{\|h\| + \|\Pi_\varepsilon h\|\} + B|h - \Pi_\varepsilon h|_1 \leq \lambda^{-n} C \|h\|$$

provided ε is chosen small enough.[18]

From the above considerations follows that $\sigma_{\text{ess}}(\mathcal{L}) \leq \lambda^{-1}$.[19] This means that the spectrum of $\mathcal{L}$ looks as in Figure 1.1. In other words out of the disc of radius λ^{-1} we have only finitely many eigenvalues of finite multiplicity.

[17] Of course if the partition has m elements then m and 1 must be considered contiguous, that is the difference in $i - j$ must be interpreted mod m.

[18] In fact, it is possible to choose the partitions of unity $\{\varphi_i\}$ such that $\|\Pi_\varepsilon\| \leq C$ independently on ε. To see it let $\{J_j\}$ be a fixed covering and choose all the $\{\varphi_i\}$ so that the $\{I_i\}$ are refinements of such a covering. Let K_j be the set of integers i such that $I_i \subset J_j$. Then,

$$\begin{aligned}\int_{S^1} \left|\sum_i \varphi_i' \frac{\int_{S^1} \varphi_i f}{\int_{S^1} \varphi_i}\right| &\leq \int_{S^1} \left|\sum_j \sum_{i\in K_j} \varphi_i' \frac{\int_{S^1} \varphi_i \int_{J_j} f}{\int_{S^1} \varphi_i |J_j|}\right| + \sum_j \sum_{i \in K_j} \int_{S^1} |\varphi_i'| \int_{J_j} |f'| \\ &\leq \sum_j \int_{S^1} |\sum_{i\in K_j} \varphi_i'| \frac{\int_{J_j} |f|}{|J_j|} + 2\int_{S^1} |f'| \\ &\leq \text{const.} \|f\|.\end{aligned}$$

[19] See [32] for a generalization of the above argument.

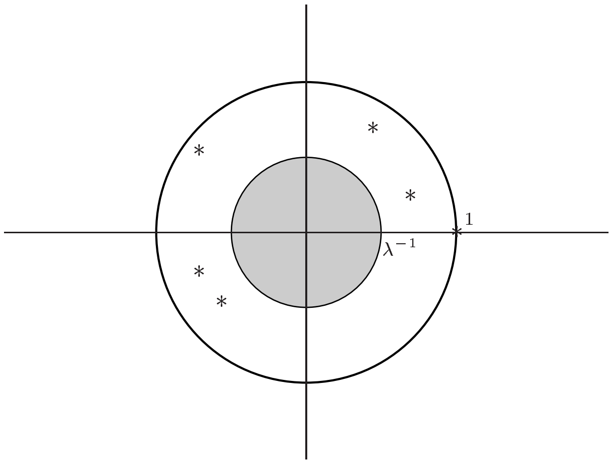

FIGURE 1.1. Spectrum of the Transfer operator

The only fact that remains to be proven is that the system is mixing: that is, 1 has multiplicity one and it is the only eigenvalue of modulus one.

First let us notice that the system is *topological mixing*, that is if A, $B \subset S^1$ are two intervals then $T^n A \cup B \neq \emptyset$ for all n sufficiently large. This property is obvious since the size of $T^n A$ is at least λ^n times the size of A, at least until $T^n A \neq S^1$.

It is then a general fact that, if the Transfer operator is quasi–compact, then topological mixing implies mixing. To see it some general considerations are called for.

To start with, due to the quasi–compactness of the operator, the following representation holds [22]:

$$\mathcal{L} = \sum_{i=1}^{N} \mu_i \Pi_i \Lambda_i + R, \tag{1.12}$$

where $|\mu_i| = 1$, Π_i are finite rank projectors such that $\Pi_i \Pi_j = \delta_{ij} \Pi_i$, $\Pi_i R = R\Pi_i = 0$, $\Pi_i N_i = N_i \Pi_i = N_i$, $\Lambda_i = \Pi_i + N_i$ and N_i are nilpotent. In addition, the spectral radius of R is strictly less than 1. Accordingly,

$$\mathcal{L}^n = \sum_{i=1}^{N} \mu_i^n \Pi_i \Lambda_i^n + R^n.$$

But, if d_i is the first integer such that $N_i^d = 0$, then

$$\Lambda_i^n = \sum_{k=0}^{d_i - 1} \binom{n}{k} N_i^k$$

Choosing f such that $\Pi_i f = f$ and $N_i f \neq 0$ but $N_i^k f = 0$ for all $k > 1$ it follows that $|\Lambda_i^n f| \geq \text{const.} n |f|$, but this is impossible since it would imply that the L^1 norm of $\mathcal{L}^n$ increases at least linearly while $\mathcal{L}$ is an L^1-contraction (see (1.6)). From these simple considerations it follows that $N_i = 0$.

Next, let us fix our attention on some projector Π_i. Since $W_{1,1} \subset L^2$ we can choose an orthonormal basis in the range of Π_i, let us call it $\{h_n\}$. Then

$$\Pi_i f = \sum_j \alpha_j(f) h_j,$$

where α_j are elements of the dual of $W_{1,1}$. In fact, it happens that the α_j are indeed measures. To see it, let us consider, for each $f \in W_{1,1} \subset C^{(0)}$

$$\begin{aligned}|\alpha_j(f)| &= \left|\int h_j \Pi_i f\right| = \left|\mu_i^{-n}\int h_j \Pi_j \mathcal{L}^n f\right| \leq |h_j|\|\Pi_j\|\|\mathcal{L}^n f\| \\ &\leq \|\Pi_j\|\{\lambda^{-n}\|f\| + B|f|\} \leq \text{const.}|f|,\end{aligned} \tag{1.13}$$

where we have used first the Lasota-York inequality (1.9) and then taken the limit $n \to \infty$.

This means that $|\alpha_j(f)| \leq$ cost $|f|_\infty$, that is α_j is an element of the dual of $C^{(0)}$, i.e. a Borel measure. Indeed, (1.13) says more: α_j is absolutely continuous with respect to the Lebesgue measure and the density belongs to L^∞. Let g_j be such density, then, for each $f \in L^1$, holds

$$\int g_j f = \alpha_j(f) = \int h_j \Pi_i f = \mu_i^{-1}\int h_j \Pi_i \mathcal{L} f = \mu_i^{-1}\int g_j \mathcal{L} f = \mu_i^{-1}\int g_j \circ T f\,,$$

which means $T_* g_j = \mu_j g_j$. Accordingly, for each $n \in \mathbb{N}$, $T_* g_j^n = \mu_j^n g_j^n$. On the other hand any eigenvalue of T_* must be an eigenvalue of $\mathcal{L}$.[20] Again quasi-compactness implies that there must be a n_j such that $\mu_j^{n_j - 1} = 1$ and $g_j^{n_j} = g_j$, otherwise there would be countably many eigenvalues of modulus one contrary to hypotheses. In other words the peripheral eigenvalues must form a finite group under multiplication.

Summing up all the previous consideration one can say that there must exists an integer $\bar{n}$ such that[21]

$$\mathcal{L}^{\bar{n}} = \Pi_* f + R_* f \tag{1.14}$$

where Π_* is a projector, R_* has spectral radius strictly less than one and $\Pi_* R_* = R_* \Pi_* = 0$. Moreover, there exists disjoint sets A_j, invariant for $T^{\bar{n}}$, and function $h_j \in W_{1,1}$ such that[22]

$$\Pi_* f = \sum_j \int_{A_j} h_j f.$$

The only obstacle to the conclusion is the fact that the sets A_j could have a complex topological structure (e.g., they could be of a Cantor-like nature). This would be reflected in the fact that their characteristic function would not be of bounded variation. In fact, if the characteristic function is of bounded variation then the set is the finite union of intervals, but then the topological mixing shows immediately that the set, to be invariant, must be all S^1. This means that the

[20] Since $\mathcal{L}$ is essentially the dual of T_*. To be more precise, it is immediate to verify that if $T_* g = \mu g$, then g is orthogonal to the range of $\mathcal{L} - \mu\text{Id}$, that is μ belongs to the spectrum of $\mathcal{L}$, but $|\mu| = 1$ then, by quasi-compactness, μ can only belong to the point spectrum.

[21] Just choose the minimum common multiple of the $n_j - 1$.

[22] Let $B_j = \operatorname{supp} g_j$, then since $g_j \circ T = g_j$, the set B_j must be invariant and it must be $g_j = \chi_{B_j}\theta_j$, where $\theta_j^{n_j - 1} = 1$. Hence θ_j can take, at most, n_j distinct values. Let A_i the sets on which θ_j is constant, clearly such sets are invariant for $T^{n_j - 1}$.

range of the projector Π_* is one, that is one is the only eigenvalue of $\mathcal{L}$ of modulus one, hence the mixing.

To see that χ_{A_j} must be of bounded variation, and thus conclude the argument, it suffices to notice that a function is of bounded variation, with bounded variation norm less or equal to one, if and only if it is in the L^1 closure of the unit ball of $W_{1,1}$.

Now let us consider some fixed j, then, by regularity of the Lebesgue measure, for each $\varepsilon > 0$ there exists $h_\varepsilon \in W_{1,1}$, $0 \leq h \leq 1$, such that

$$\int |\chi_{A_j} - h_\varepsilon| \leq \varepsilon.$$

First remark that

$$\mathcal{L}^{\bar{n}} h_j \chi_{A_j} = \mathcal{L}^{\bar{n}} h_j \chi_{A_j} \circ T = \chi_{A_j} \mathcal{L}^{\bar{n}} h_j = \chi_{A_j} h_j,$$

then notice

$$\|\mathcal{L}^{k\bar{n}} h_j h_\varepsilon\| \leq \lambda^{-k\bar{n}} \|h_j h_\varepsilon\| + B|h_j|_\infty |h_\varepsilon|.$$

Using the above two inequalities, on the one hand we have

$$|h_j \chi_{A_j} - \mathcal{L}^{k\bar{n}} h_j h_\varepsilon| \leq |h_j|_\infty \varepsilon$$

on the other hand, by choosing k_ε so large that $\|\mathcal{L}^{k_\varepsilon \bar{n}} h_j h_\varepsilon\| \leq 2B|h_j|$, we are approximating χ_{A_j} by a sequence of functions uniformly bounded in $W_{1,1}$. This, as already remarked implies that χ_{A_j} is of bounded variation.

1.3.1. *Conclusion.* The results of the above discussion can be summarized in the following well know result.

Theorem 1.1. *If $T : S^1 \to S^1$ is twice differentiable and $|DT| > \lambda > 1$, then there exists a unique invariant measure μ_* absolutely continuous with respect to Lebesgue, moreover $h_* := \frac{d\mu_*}{dm} \in W_{1,1}$. The dynamical system (S^1, T, μ_*) is mixing. In addition, there exists $\Lambda \in (0,1)$ such that for all measures μ absolutely continuous with respect to m, such that $h := \frac{d\mu}{dm} \in W_{1,1}$, holds*

$$|\mu(T_*^n f) - \mu_*(T_*^n f)| \leq const. \Lambda^n \|h - h_*\| \, |f|.$$

The above theorem is certainly very strong and quite satisfactory. The only reason for unhappiness that I can see is the lack of any explicit knowledge on the constant Λ. We will see how to overcome even this limitation in the next lecture.

2. Second Lecture

In this lecture we continue our investigation of efficient methods to establish the convergence to equilibrium by treating explicitly the simple example of smooth expanding circle maps.

2.1. **A geometrical approach (Projective metrics).** Projective metrics are widely used in geometry, not to mention the importance of their generalizations (e.g. Kobayashi metrics) for the study of complex manifolds [48]. It is quite surprising that they play a major rôle also in our situation, [51].

Let us start with few word on the Hilbert metric, a quite important tool in hyperbolic geometry.

2.1.1. *Projective metrics.* Let $C \in \mathbb{R}^n$ be a strictly convex compact set. For each two point $x, y \in C$ consider the line $\ell = \{\lambda x + (1 - \lambda y) \mid \lambda \in \mathbb{R}\}$ passing through x and y. Let $\{u, v\} = \partial C \cap \ell$ and define[23]

$$\Theta(x, y) = \left| \ln \frac{\|x - u\| \|y - v\|}{\|x - v\| \|y - v\|} \right|$$

(the logarithm of the cross ratio). By remembering that the cross ratio is a projective invariant and looking at Figure 2.2 it is easy to check that Θ is indeed a metric. Moreover the distance of an inner point from the boundary is always infinite. One can also check that if the convex set is a disc then the disc with the Hilbert metric is nothing else than the Poincaré disc.

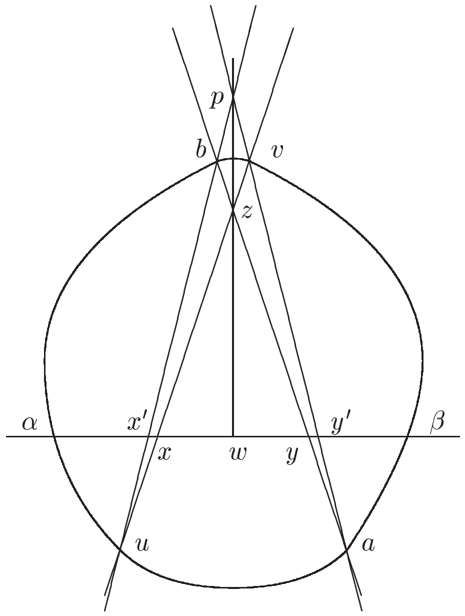

FIGURE 2.2. Hilbert metric

The object that we will use in our subsequent discussion are not convex sets but rather convex cones, yet their projectivization is a convex set and one can define the Hilbert metric on it (whereby obtaining a semi-metric for the original cone). It turns out that there exists a more algebraic way of defining such a metric, which is easier to use in our context. Moreover, there exists a simple connection between vector spaces with a convex cone and vector lattices (in a vector lattice one can always consider the positive cone). This justifies the next digression in lattice theory.[24]

[23]Remark that u, v can also be ∞.
[24]For more details see [7], and [65] for a recent overview of the field.

Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[25] We require the partial order to be *continuous*, i.e. given $\{v_n\} \in \mathbb{V}$ $\lim_{n\to\infty} v_n = v$, if $v_n \succeq w$ for each n, then $v \succeq w$.

We define the closed convex cone[26] $\mathcal{C} = \{v \in \mathbb{V} \mid v \neq 0,\, v \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed). Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$v \preceq w \iff w - v \in \mathcal{C} \cup \{0\}.$$

It is then possible to define a projective metric Θ (Hilbert metric),[27] in $\mathcal{C}$, by the construction:

$$\begin{aligned}
\alpha(v,\, w) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda v \preceq w\} \\
\beta(v,\, w) &= \inf\{\mu \in \mathbb{R}^+ \mid w \preceq \mu v\} \\
\Theta(v,\, w) &= \log\left[\frac{\beta(v,\, w)}{\alpha(v,\, w)}\right]
\end{aligned} \tag{2.1}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

The importance of the previous constructions is due, in our context, to the following theorem by G.Birkhoff [8].

Theorem 2.1. *Let $\mathbb{V}$ a topological vector space; $L : \mathbb{V} \to \mathbb{V}$ a continuous linear map such that $L(\mathcal{C}) \subset \mathcal{C}$, for a closed convex cone $\mathcal{C} \subset \mathbb{V}$ with $\mathcal{C} \cap -\mathcal{C} = \emptyset$. Let Θ be the Hilbert metric corresponding to the cone $\mathcal{C}$. Setting $\Delta = \sup_{v,\, w \in L(\mathcal{C})} \Theta(v,\, w)$ we have*

$$\Theta(Lv,\, Lw) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta(v,\, w) \qquad \forall v,\, w \in \mathcal{C}$$

($\tanh(\infty) \equiv 1$).

Remark 2.1. *In general, it suffices to know that $L(\mathcal{C}) \subset \mathcal{C}$ in order to conclude $\Theta(Lv,\, Lw) \leq \Theta(v,\, w)$. However, a strict contraction depends on the diameter of the image being finite.*

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of L (see [7]).

Usually the space $\mathbb{V}$ comes endowed with its own metric, in such a case it is natural to wonder about the strength of the Hilbert metric compared to such a metric. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting general result.

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[25] We are assuming the partial order to be well behaved with respect to the algebraic structure: for each $v,\, w \in \mathbb{V}$ $v \succeq w \iff v - w \succeq 0$; for each $v \in \mathbb{V}$, $\lambda \in \mathbb{R}^+\backslash\{0\}$ $v \succeq 0 \implies \lambda f \succeq 0$; for each $v \in \mathbb{V}$ $v \succeq 0$ and $v \preceq 0$ imply $v = 0$ (antisymmetry of the order relation).

[26] Here, by "cone," we mean any set such that, if v belongs to the set, then λv belongs to it as well, for each $\lambda > 0$.

[27] In fact, we define a semi–metric, since $v = \lambda w \Rightarrow \Theta(v,\, w) = 0$. As already mentioned, the metric that we describe corresponds to the conventional Hilbert metric on the projectivization $\widetilde{\mathcal{C}}$ of the cone $\mathcal{C}$.

Definition 2. *A function $\rho : \mathbb{V} \to \mathbb{R}^+$ is called* homogeneous of degree one *if for all $\lambda \in \mathbb{R}$ and $v \in \mathbb{V}$*

$$\rho(\lambda v) = |\lambda| \rho(v).$$

Remark 2.2. *Note that a norm or a linear functional are both homogeneous function of degree one.*

Definition 3. *A homogeneous function of degree one is called* adapted to a cone $\mathcal{C}$ *if, for each $v, w \in \mathbb{V}$,*

$$-v \preceq w \preceq v \Longrightarrow \rho(v) \geq \rho(w),$$

and $v \in int\, \mathcal{C}$ implies $\rho(v) > 0$.

Lemma 2.1. *Let ρ_i be two homogeneous functions of degree one adapted to the cone $\mathcal{C} \subset \mathbb{V}$. Then, given v, $w \in int\, \mathcal{C} \subset \mathbb{V}$ for which $\rho_1(v) = \rho_1(w)$,*

$$\rho_2(v - w) \leq \left(e^{\Theta(v,w)} - 1\right) \min\{\rho_2(v), \rho_2(w)\}.$$

Typically in the applications ρ_2 will be a norm while ρ_1 some functional (in our case the integration with respect to Lebesgue measure). Many normed vector lattices satisfy the hypothesis of Lemma 1.3 (e.g. Banach lattices[28]).

2.1.2. *Application to expanding maps.* Let us finally get down to business and see how all the above machinery can be used to investigate the return to equilibrium in our toy example.

Consider a dynamical partition $\mathcal{P}_m$.[29] Let us define the convex cone

$$\mathcal{C}_{a,m} = \left\{ f \in C^{(0)}(S^1) \;\middle|\; \mathbb{E}(f \mid \mathcal{F}_m) \geq 0;\ |f'|_1 \leq a \int_{S^1} f \right\}. \tag{2.2}$$

Where $\mathcal{F}_m$ is the σ-algebra generated by the partition $\mathcal{P}_m$ and $\mathbb{E}(\cdot \mid \mathcal{F}_m)$ is the conditional expectation with respect to the Lebesgue measure.

The fist relevant fact consists in the following computation[30]

$$\begin{aligned}|(\mathcal{L}f)'|_1 &\leq \lambda^{-1}|f'|_1 + B|f|_1 \leq \lambda^{-1}|f'|_1 + B\int_{S^1} f + B|f - \mathbb{E}(f \mid \mathcal{F}_m)|_1 \\ &\leq (\lambda^{-1} + B\lambda^{-m})|f'|_1 + B\int_{S^1} f.\end{aligned}$$

The above means that if $\nu \in (\lambda^{-1}, 1)$, $\lambda^{-m} \leq \frac{\nu - \lambda^{-1}}{B}$ and $a = 2B(1-\nu)^{-1}$, then $\mathcal{L}f$ satisfies the second condition defining the cone. What about the first conditions?

$$\mathcal{L}^m f(x) \geq \sum_{y \in T^{-m}x} |D_y T^m|^{-1} \left\{ \mathbb{E}(f \mid \mathcal{F}_m)(y) - \int_{I(y)} |f'(\xi)| d\xi \right\}$$

[28] A Banach lattice $\mathbb{V}$ is a vector lattice equipped with a norm satisfying the property $\| \, |v| \, \| = \|v\|$ for each $f \in \mathbb{V}$, where $|v|$ is the least upper bound of v and $-v$. For this definition to make sense it is necessary to require that $\mathbb{V}$ is "directed," i.e. any two elements have an upper bound.

[29] By *dynamical partition* we mean a partition obtained in the following way. Let $\mathcal{P}_0$ be any partition such that T restricted on its element is one to one and the image of each of its elements is the all circle. Then $\mathcal{P}_m$ is nothing else than $T^{-m}\mathcal{P}_0$. It should be remarked that any sufficiently fine partition would do (see [52]) and even a smooth partition of unity could be used. The special choice here is determined only by didactic reasons.

[30] Remember inequality (1.7).

where $I(y)$ is the element of $\mathcal{P}_m$ which contains the point y. To continue it is necessary a standard type of argument in hyperbolic theory: a distortion estimate.

$$\begin{aligned}1 &= \int_{S^1} dx = \int_{I(y)} |D_\xi T^m| d\xi \geq \int_{I(y)} |D_y T^m| e^{-\sum_{i=1}^m |\ln |D_{T^i\xi}T| - \ln |D_{T^i y}T||} \\ &\geq |I(y)||D_y T^m| e^{-(1-\lambda^{-1})^{-1}B_0}\end{aligned}$$

Let $D := e^{-(1-\lambda^{-1})^{-1}B_0}$, then the above equations yield

$$\mathcal{L}^m f(x) \geq \sum_{I\in\mathcal{P}_m} \left\{ D\int_I f - \lambda^{-m}\int_I |f'| \right\} \geq (D - \lambda^{-m}a)\int_{S^1} f \geq \frac{D}{2}\int_{S^1} f \tag{2.3}$$

provided we choose m so that $D - 2\lambda^{-m}a > 0$.

This means that, by choosing m such that $\lambda^{-m} \leq \frac{D}{4B}(1-\nu)$, holds $\mathcal{L}^m\mathcal{C}_{a,m} \subset \mathcal{C}_{\sigma a,m}$ with $\sigma = \frac{1+\nu}{2} < 1$. In addition, it is easy to compute that[31]

$$\Delta := \operatorname{diam}(\mathcal{L}^m\mathcal{C}_{a,m}) \leq 2\ln\left[2\frac{1+B+2(1+D)\frac{\nu-\lambda^{-1}}{1-\nu}}{D}\right] := 2\ln\delta < \infty. \tag{2.4}$$

The estimate (2.4) can be used together with Theorem 2.1 and Lemma 2.1 to prove:

Theorem 2.2. *If $T : S^1 \to S^1$ is twice differentiable and $|DT| > \lambda > 1$, then there exists a unique invariant measure μ_* absolutely continuous with respect to Lebesgue, moreover $h_* := \frac{d\mu_*}{dm} \in W_{1,1}$. The dynamical systems (S^1, T, μ_*) is mixing. In addition, there exists $\Lambda \in (0,1)$ such that for all measures μ absolutely continuous with respect to m, such that $h := \frac{d\mu}{dm} \in W_{1,1}$, holds*

$$|\mu(T_*^n f) - \mu_*(T_*^n f)| \leq const.\Lambda^n \|h - h_*\|\, |f|.$$

In addition,

$$\Lambda \leq \left[\tanh\frac{\Delta}{4}\right]^{\frac{1}{m}} = \left[\frac{\delta+1}{\delta-1}\right]^{\frac{1}{m}}.$$

Note that the bound for the contraction rate Λ it is now explicitly computable. We do not insist on its actual value since the above bound is too simplistic to be optimal, the goal here was only to emphasize the possibility to obtain explicit bounds.[32]

2.2. **A probabilistic approach (coupling).** There exists another general approach which yields constructive bounds on the rate of relaxation to equilibrium. This point of view has a strong probabilistic flavor. Nevertheless, although not immediately algebraic as the previous ones, it can be phrased at the level of the observable and the measures as well, so it is no inconceivable that it may have a relevance for quantum systems as.

Before describing this method let us say two words on coupling in general.

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[31] To see it compute the distance of a generic element $f \in \mathcal{C}_{a,m}$ from 1. This is done by looking for λ, μ such that $\lambda \preceq f \preceq \mu$. This immediately yields $\lambda \leq \min\{\inf \mathbb{E}(f \mid \mathcal{F});\ \int_{S^1} f - \frac{1}{a}|f'|_1\}$ and $\mu \geq \max\{\sup \mathbb{E}(f \mid \mathcal{F});\ \int_{S^1} f + \frac{1}{a}|f'|_1\}$. Now if $f \in \mathcal{L}^m\mathcal{C}_{a,m}$, according to the above discussion follows $\lambda \leq \min\{\frac{D}{2};\ 1-\sigma\}\int_{S^1} f := \alpha$ and $\mu \geq \max\{1 + a\lambda^{-m} + B + D\lambda^{-m}a;\ 1+\sigma\}\int_{S^1} f := \beta$. Thus the distance between f and 1 is given by $\ln\frac{\beta}{\alpha}$. The diameter is then obviously less than twice such a distance.

[32] For the reader interested in sharp bounds see, e.g., [6], [43], [53].

2.2.1. *Coupling.* Let us think of discrete random variables for one moment. A coupling is a way of filling an array with a constraint on the sums of the lines and of the columns. Given two distributions $(\mu_i)_{i=1,\dots,n}$ and $(\nu_i)_{i=1,\dots,n}$, it is an array $(p_{i,j})_{i,j=1,\dots,n}$ with non negative entries satisfying $\sum_j p_{i,j} = \mu_i$ and $\sum_i p_{i,j} = \nu_j$. Let us see how to construct a coupling with a strong correlated part. We can fill first the diagonal of the array in an arbitrary way imposing only that the value of each entry is less than the constrained values of the associated row and column (this will be the "correlated part"). For example choose a third distribution $(\eta_i)_{i=1,\dots,n}$ and an $\varepsilon > 0$ small enough so that for all i, $\varepsilon\eta_i < \min\{\mu_i, \nu_i\}$ and put the values $\varepsilon\eta_i$ on the diagonal.[33] The total amount of mass "used" is ε. The remainder can (among other possibilities) be coupled "independently",

$$p_{i,j} = \chi_{\{i=j\}}\varepsilon\eta_i + (1-\varepsilon)^{-1}(\mu_i - \eta_i)(\nu_j - \eta_j).$$

Such an array is indeed a coupling of μ and ν. One can realize this coupling as follows. Drop a Bernoulli coin $(\varepsilon, 1-\varepsilon)$. If you get head, then choose the two random variables according to the distribution η (in this case they take the same value). If you get tail, then choose both random variables independently according to the "remaining" joint distribution, $(1-\varepsilon)^{-2}(\mu_i - \eta_i)(\nu_j - \eta_j)$.

This last point of view easily generalizes to the case of continuous random variables with densities. Let X and Y be two random variables valued in S^1 distributed according to smooth densities with respect to the natural measure on S^1. More precisely, let hdm be the distribution of X and gdm the distribution of Y. A moment of thought shows that T^nX and T^nY are distributed according to $\mathcal{L}^n hdm$ and $\mathcal{L}^n gdm$, respectively. We have thus a clear connection with the language of the previous approach.

In this context our goal will be to construct a coupling $(\tilde{X}, \tilde{Y})$ of these random variables such that, in mean, the distance between $T^n\tilde{X}$ and $T^n\tilde{Y}$ decays to 0. Since the dynamic is deterministic, the only possibility is to couple the initial distributions.

In our case a coupling of two random variables is a joint distribution, that is a distribution on $S^1 \times S^1$ with the marginals given by the two random variables. The simpler example is the so-called *independent coupling* (simply take the product distribution). If the random variables have the same law, another simple coupling is the *diagonal coupling* ($\tilde{X} = \tilde{Y}$ with probability one). Intermediate cases present some "correlation" between the two random variables.

Here is an easy way to construct other coupling in the case in which both h and g are bonded away from zero. Consider a density function f and $\varepsilon > 0$ such that $\varepsilon f(x) < \inf\{g(x), h(x)\}$. Now, drop a Bernoulli coin distributed according to $(\varepsilon, 1-\varepsilon)$. If you obtain head, then choose the two random variables equal and according to the distribution with density f; if the result is tail, then choose both random variables independently according to the "remaining" joint distribution, whose density on the square is given by $F(x,y) = (1-\varepsilon)^{-2}(g(x) - \varepsilon f(x))(h(y) - \varepsilon f(y))$.

It should be intuitive that with the above distribution the two random variable are "closer" than with respect to the independent coupling. To make this precise we need another very helpful fact regarding coupling: the possibility to use it to

[33] To be optimal, one could choose η so that $\varepsilon\eta_i = \min\{\mu_i, \nu_i\}$. But then, the remainder cannot be coupled independently.

define a metric in the space of measures. Let us μ, ν be two measures on S^1 and ρ a distance. If $C_{\mu,\nu}$ is the collection of all the couplings of μ, ν then we can define the Vasserstain distance[34] by

$$\bar{d}(\mu,\nu) := \inf_{\tilde{\mu}\in C_{\mu,\nu}} \tilde{\mu}(\rho). \tag{2.5}$$

Back to our case, we will choose

$$\rho(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases} \tag{2.6}$$

It is easy to see that the convergence in the norm determined by (2.5) implies strong convergence for the measure. In fact,

$$\left|\int hf - \int gf\right| \leq \int |f(x) - f(y)|\tilde{\mu}(dx, dy) \leq 2\|f\|_\infty \int \rho(x,y)\tilde{\mu}(dx,dy)$$

and taking the minimum on the coupling the result follows.

2.2.2. *Application to expanding maps.* The aim is then clearly to construct explicitly a coupling with a lot of mass concentrate on the diagonal in order to show that the measures converge.

Let us consider two densities h, g and let μ, ν be the associated probability measures. We will construct couplings of the type[35]

$$d\tilde{\mu} = (1-\varepsilon)\varphi_h(x)\varphi_g(y)dxdy + \varepsilon\psi(x)\delta(x-y)dxdy \tag{2.7}$$

The convenience of the above coupling is that, if we define the operator

$$\mathcal{L}_2\tilde{\mu}(f) := \tilde{\mu}(f \circ T \times T)$$

then $\mathcal{L}_2\tilde{\mu}$ is a coupling of $T^*\mu$, $T^*\nu$.[36] It is easy to see that the density of $\mathcal{L}_2\tilde{\mu}$ is given by

$$d\mathcal{L}_2\tilde{\mu} = (1-\varepsilon)\mathcal{L}\varphi_h(x)\mathcal{L}\varphi_g(y)dxdy + \varepsilon\mathcal{L}\psi(x)\delta(x-y)dxdy$$

which means that the coupling of the two random variable in time remains the same: a singular part and a product part.

We are in position to describe the strategy of the argument.

Let us start by considering the invariant set

$$\mathcal{C}_a := \left\{ h \in C^{(1)} \;\middle|\; h \geq 0;\ \int_{S^1} h = 1;\ \frac{h(x)}{h(y)} \leq e^{a|x-y|} \right\} \tag{2.8}$$

By computations similar to the ones in section 1.3, it follows that if $\nu \in (\lambda^{-1}, 1)$ and $a = \frac{B_0}{\nu - \lambda^{-1}}$ then $\mathcal{L}\mathcal{C}_a \subset \mathcal{C}_{\nu a}$. It is also immediately clear that, if $h \in \mathcal{C}_a$, then $\inf f \geq \alpha := e^{-a}$.

Next, let us start with two densities $h, g \in \mathcal{C}_{a,m}$ as in the previous section. We can then perform the coupling (2.7) with $\varepsilon = \frac{\alpha}{2}$ $\psi = 1$, $\varphi_h := \frac{h-\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$ and $\varphi_g := \frac{g-\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$. Consequently, we have that the non diagonal part of the coupling is determined is

[34] Sometime it is called the $\bar{d}$ metric.

[35] By $\delta(x)$ we mean, as usual, the 'density' of the probability measure with mass 1 in zero.

[36] That is of the measures with densities $\mathcal{L}h$, $\mathcal{L}g$ respectively.

determined by two densities (since the functions are still positive) that belongs to $\mathcal{C}_{2a}$ since

$$\begin{aligned}\frac{h(x)-\frac{\alpha}{2}}{h(y)-\frac{\alpha}{2}} &\leq \frac{e^{a|x-y|}h(y)-\frac{\alpha}{2}e^{a|x-y|}+\frac{\alpha}{2}(e^{a|x-y|}-1)}{h(y)-\frac{\alpha}{2}}\\ &\leq e^{a|x-y|}(2-e^{-a|x-y|})\leq e^{a|x-y|}(1+a|x-y|)\\ &\leq e^{2a|x-y|}.\end{aligned}$$

Accordingly, if k is the first integer such that $2\nu^k\leq 1$, it follows that $h_1:=\mathcal{L}^k\varphi_h\in\mathcal{C}_a$ and $g_1:=\mathcal{L}^k\varphi_g\in\mathcal{C}_a$. This means that the two densities h_1 and h_2 can be coupled again. Doing this over and over, after a time n we couple a mass of the order of $(1-\frac{\alpha}{2})^{\frac{n}{k}}$. This means that we can prove a result similar to the previous ones.

Theorem 2.3. *If $T:S^1\to S^1$ is twice differentiable and $|DT|>\lambda>1$, then there exists a unique invariant measure μ_* absolutely continuous with respect to Lebesgue, moreover $h_*:=\frac{d\mu_*}{dm}\in W_{1,1}$. The dynamical systems (S^1,T,μ_*) is mixing. Moreover, there exists $\Lambda\in(0,1)$ such that for all measures μ absolutely continuous with respect to m, such that $h:=\frac{d\mu}{dm}\in C^{(1)}$, holds*

$$|\mu(T_*^n f)-\mu_*(T_*^n f)|\leq const.\Lambda^n\{|h'-h_*'|_\infty+|h-h_*|_\infty\,|f|_\infty\}.$$

In addition,

$$\Lambda\leq(1-\frac{\alpha}{2})^{\frac{n}{k}}.$$

Also in this case we have thus again an explicit estimate of the rate of convergence to equilibrium. The bound is of the same type of the previous one although k is smaller than m so we have a sharper result. At any rare, as in the previous section, we did not push the bound to optimality to let the reader appreciate better the simplicity of this approach. It must be noted that the space of densities is slightly different from the previous ones, this is done only to simplify the argument. Another, relevant comment is that in the bound it appears the norm $|\cdot|_\infty$, rather than the norm $|\cdot|_1$, of f. This is clearly a weaker result, it is the price to pay for a simpler argument and, possibly, a better numerical bound. Nevertheless, there exist several practical cases where the difference in norms does matter.

3. Third Lecture

In the previous lecture we have illustrated, via the study of a toy example, some techniques that have yielded important results in the classical theory of dynamical systems as far as convergence to equilibrium is concerned. Yet, the reader can be left with the impression that the above techniques are very special and have no applicability beyond the mentioned example. To fight this perception we will briefly discuss more general cases.

3.1. **Anosov systems.** Anosov systems are systems such that at each point the tangent space can be split in two subspaces. Vectors in one subspace (the unstable one) are uniformly expanded by the dynamics, while vectors in the other (the stable one) are uniformly contracted. In addition the splitting is required to be invariant and continuous.[37]

[37]By general results this implies that the space can be foliated in manifolds with the same properties. Points in the same stable manifold get exponentially closer in the future, while points

For simplicity keep in mind the prototype of such systems, the so called *Arnold cat*:

$$T(x, y) := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \mod 1.$$

To study the above class of systems it is possible to proceed in analogy with what we have done in the previous lectures. First of all, let us define a cone of densities. Since the natural invariant measures for Anosov systems may be non absolutely continuous with respect to Lebesgue[38] we are forced to consider not just function but generalize function (i.e. *distributions*). It is then natural to start by defining a set of test functions:

$$\mathcal{D}_b := \left\{ \varphi \in C^{(1)} \;\middle|\; \varphi \geq 0;\ \|D^s \varphi\|_\infty \leq b \int \varphi \right\}, \tag{3.1}$$

where we use D^s, D^u to designate the stable and unstable derivative respectively.[39] We consider then the following cone[40]

$$\mathcal{C}_{a,b} = \left\{ h \in C^{(1)} \;\middle|\; \|D^u h\|_\infty \leq a \int h;\ \int h\varphi \geq 0\ \forall \varphi \in \mathcal{D}_b \right\} \tag{3.2}$$

To the above cone is 'naturally' associated an adapted norm

$$\|h\| := \|D^u h\|_\infty + b \sup_{\|D^s \varphi\|_\infty \leq 1} \int h\varphi \tag{3.3}$$

It turns out that, considering the Banach space B obtained by closing $C^{(1)}$ with respect to the norm (3.3), the transfer operator (that in this case, since each point has only one preimage and the Lebesgue measure is preserved, it is nothing else but the composition with T^{-1}) has spectral radius one. In addition, one is an isolated eigenvalue with multiplicity one, the essential spectrum is contained in a smaller circle (determined by the minimal expansion rate) and all the eigenvalue outside the essential spectrum have modulus strictly less than one. In conclusion, for this type of system holds exactly the same spectral picture given in figure 1.1.

3.2. **About infinite systems.** As we have remarked at the beginning, if we want to treat quantum systems, then we need to investigate infinite systems. It is then clear that what we have discussed so far cannot be satisfactory from this point of view since we have always considered systems with very low dimension. The next question is then: what is know, classically, about infinite systems?

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in the same unstable manifold becomes exponentially closer in the past. This is a more general setting for the above mentioned phenomena of instability. The reader should not be mislead by he presence of stable subspaces: taken two points at random they will almost surely not belong to the same stable sub-manifold, hence they will tent to have drastically different evolutions: dynamical instability.

[38]In the case at hand the Lebesgue measure is obviously invariant, yet there are many simple examples where the invariant measure must be singular. For example consider the solid torus $M = S^1 \times D^2$, where D^2 is the unit disk in $\mathbb{R}^2$. Consider a map that stretches the torus to twice its length shrinks it transversally and wrap it around twice. Such a type of maps are called *solenoid* and a concrete analytic expressions of a map $T : M \to M$ of this type is $T(\varphi, x, y) = (2\varphi, \frac{1}{10}x + \frac{1}{2}\cos\varphi, \frac{1}{10}y + \frac{1}{2}\sin\varphi)$, [42]. It is easy to check that the map is one one but it shrinks the volume, so the invariant measure must be supported on a set of zero Lebesgue measure.

[39]By this we mean the derivative taken along the stable or the unstable direction

[40]The cone is a bit simplified for the case at hand, yet with few technical changes it is suitable for treating general Anosov system[11].

Essentially the only class for systems that has been investigate with some success, although much still remains to be done, are the *coupled maps lattice*. Such systems can be thought of as a collection of infinitely many identical systems (on a lattice) that interact weakly. To be more precise the X be the phase space of a single systems and $T : X \to X$ the relative dynamics. For simplicity suppose that the lattice is simply $\mathbb{Z}$, then the phase space of the all system is given by $X^{\mathbb{Z}}$ while the dynamics is defined as

$$(\tilde{T}_\varepsilon x)_i := Tx_i + \varepsilon\Psi(Tx) \tag{3.4}$$

where $\mathrm{Id} + \varepsilon\Psi : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ is one to one, onto and smooth.

The easiest case is when the single site space is S^1 and the relative map nothing else than the expanding maps that we have already investigated.

On the above systems there exist a manifolds of results starting from the seminal work of Bunimovich and Sinai [18] to the ones of Bricmont and Kupianen [9], [10] where a powerful technique based on the cluster expansion is developed (see also [60]).

The strongest results can be found in [5] and [25] where the transfer operator is studied on suitable Banach spaces and, using the above mentioned cluster expansion technique, it is proven the existence of a unique SRB measure and the exponential convergence to equilibrium, including spatial mixing. Some more general results are found in [68] and [38] where the theory is extended to coupling of higher dimensional maps. For a general review on couple map lattices see [39].

The problem with the above approach is that it is limited to smooth cases, for example it does not cover the most natural nearest neighborhood coupling

$$\Psi(x)_i = x_{i-1} - 2x_i + x_{i+1},$$

for which the property of being onto fails.[41]

There is clearly the need of more general (and more akin to what we have seen so far) techniques to study such systems. In this direction the only relevant results are due to Keller and Künzle [47] and Keller [45], [46]. Unfortunately they are only able to prove, for the more general systems, the existence of the SRB measure and not its uniqueness, let alone the mixing properties.

3.3. **Ergodicity of infinite systems.** At this point a word is in order on the problem of the ergodicity on infinite systems (in the situation above, the uniqueness of the SRB measure). In fact, it is not obvious what we should mean with the sentence: such and such system is *ergodic*.

To appreciate the problem let us consider a free gas or an infinite chain of harmonic oscillators, or a systems of hard rods. All the above mentioned systems are ergodic, indeed Bernoulli, with respect to any Gibbs state ([74], [67], [49]). Is it reasonable to call them ergodic?

My resistance stems from the fact that all such systems are integrable when the number of degree of freedom in finite and the ergodicity of the infinite systems arises essentially from the fact that all the alternative invariant measures become mutually singular in the thermodynamics limit.

[41] In fact, when the single map is one dimensional expanding, such a coupling is either not smooth (if $X = S^1$) or not onto (if $X = [0, 1]$)).

More precisely, when we say that a finite system is ergodic we mean that there is only one measure absolutely continuous with respect to Lebesgue (or, more generally, only one SRB measure).[42] When the system becomes infinite the rôle of the Lebesgue measure (for volume preserving dynamics) is taken by the Gibbs measures. But the class of measures absolutely continuous with respect to a given Gibbs measure is a rather small class for infinite systems, in particular all the Gibbs states at different temperatures are mutually singular.

To have a better idea of this phenomena let us consider a box containing particles and compare the measure for which the particles are uniformly distributed in the box with the one for which they are uniformly distributed only in half of the box. If the box is $[0,1]^3$ and m is the Lebesgue measure then the configuration space is given by $[0,1]^{3N}$ where N is the number of particles and the uniform distribution corresponds to $\mu_N := \otimes_{i=1}^N m$. If g is the characteristic function of the region $\Lambda := [0,\frac{1}{2}] \times [0,1]^2$, then the uniform distribution in such a box is given by the measure $\hat{\mu}_N$ defined by

$$\hat{\mu}_N(f) := \frac{1}{\mu_N(\Lambda^N)} \int_{[0,1]^{3N}} \prod_{i=1}^N g(x_i) f(x_1,\dots,x_N)\mu_N(dx_1,\dots,dx_N). \tag{3.5}$$

Clearly, if the number of degrees of freedom becomes infinite the measure $\hat{\mu}_N$ becomes singular with respect to the measure μ_N since the relative density explodes. It is then clear that to talk about ergodicity in infinite dimension in a satisfactory way it is necessary to define a priory the class of measures in which we want the unicity (see [29] for similar considerations). The following proposal is essentially due to Sinai.

Definition 4. *Let $\mathcal{M}$ be a class of measures that contains all the Gibbs measures, then we call a systems $\mathcal{M}$-ergodic if each $\mu \in \mathcal{M}$, μ invariant, is a convex combination of Gibbs measures.*

The definition of the class of measures is clearly vastly arbitrary and it can be justified only a posteriori via the eventual successes of the theory, nevertheless it seems natural to consider the marginals μ_Λ, where Λ is some finite region, and to require that $\frac{d\mu_\Lambda}{dm}$ grows in a controlled manner with $|\Lambda|$. A measure of such a growth that has proven useful in statistical mechanics and hydrodynamics is the *specific entropy*

$$H_\Lambda(Q \mid P) = \sup_{\varphi \in \mathcal{F}_\Lambda} \{\mathbb{E}^Q(\varphi) - \ln \mathbb{E}^P(e^\varphi)\} \tag{3.6}$$

where by $\mathbb{E}^Q(f)$ we mean the expectation of f wit respect to Q, that is $\int f(\omega)dQ(\omega)$.

Definition 5. *Given a Gibbs measure P we say that a measure Q has finite specific entropy if*

$$H_\Lambda(Q \mid P) \le C(1 + |\Lambda|)$$

for each cubic box Λ.

---

[42]For finite systems we mean that it has only one measure "related" to the Lebesgue measure (that is, only one SRB measure). This can be made precise in many ways, probably the simplest one is to consider only invariant measures whose conditional expectation with respect to the unstable foliation be absolutely continuous with respect to the restriction of the Lebesgue measure on almost every leave. Clearly this excludes pathological cases, like invariant measures supported by a periodic orbit, which always exists but are not very relevant for the statistical properties of the system.

Let the two measures Q_Λ and P_Λ be the restrictions to Q and P respectively to the region Λ. If Q_Λ is absolutely continuous with respect to P_Λ (i.e., $\frac{dQ_\Lambda}{dP_\Lambda} = \rho_\Lambda$) with density $\rho_\Lambda \in L^\infty$ then an easy computation shows that $H_\Lambda(Q \mid P) = \int \rho_\Lambda \ln \rho_\Lambda dP_\Lambda$. In fact, it turns out that if two measure have finite relative entropy, then they are absolutely continuous.

If we go back to our previous example of the particles in a box versus the particles in half box then we can compute the relative entropy of the two measures since the density of $\hat\mu_{\mathcal{M}}$ with respect to μ_N is given by (3.5). We obtain $H(\hat\mu_N \mid \mu_N) = N \ln 2$. We see now very well the phenomena of the two measure becoming singular when the number of particles grows: it is exactly quantified by the divergence of the relative entropy.

A reasonable proposal is then to choose the class $\mathcal{M}$ to be the class of measures of *finite specific entropy*. In my knowledge couple map lattices are the only example where ergodicity in the above sense is established [5], [25]. It is then fair to say that nothing much can be said for deterministic systems, but a little noise may help.

3.4. **Noisy systems.** In the above context a model relevant to the previous discussion has been introduced by Olla, Varadhan and Yau [66]. The model consists of infinitely many identical particles (we call ω the phase space of locally finite[43] configurations $\omega = (q_i, p_i)$, $q_i, p_i \in \mathbb{R}^3$) undergoing a motion determined by an Hamiltonian

$$H(q,p) = \sum_i \phi(p_i) + \frac{1}{2}\sum_{ij} V(q_i - q_j),$$

ϕ being a strictly convex function with at most linear growth.[44] Let L be the associated Liouville operator.[45] In addition, there is a long range stochastic interaction among particles. The equations defining the motion read

$$\begin{cases} dq_\alpha = \phi'(p_\alpha)\, dt \\ dp_\alpha = -\sum\limits_{\beta \neq \alpha} V'(q_\alpha - q_\beta)\, dt + \varepsilon b_\alpha(\omega)\, dt + \varepsilon \sum\limits_{\substack{\theta \in \{1,\dots,d\} \\ \beta \neq \alpha}} \sigma^\theta_{\alpha,\beta}(\omega)\, dw^\theta_{\alpha,\beta}, \end{cases} \tag{3.7}$$

where $w^\theta_{\alpha,\beta}$ is a family of independent one-dimensional Wiener processes for $\theta = 1, 2, ..., d$ and $\alpha \neq \beta$ such that $w^\theta_{\alpha,\beta} = -w^\theta_{\beta,\alpha}$; $\varepsilon \in \mathbb{R}^+$; ϕ' and V' denote the gradient of ϕ and V, respectively. The coefficients $b_\alpha, \sigma^\theta_{\alpha,\beta} : \Omega \mapsto \mathbb{R}^3$ are appropriate smooth local functions, chosen in such a way that total energy and momentum are both preserved by the randomized evolution (3.7).[46]

Let the operator $\hat L$ be the generator of the stochastic part of the dynamics, it reads

$$\widehat{L}\psi = \sum_{\alpha \in I} \langle b_\alpha, \frac{\partial \psi}{\partial p_\alpha} \rangle + \frac{1}{4} \sum_{\theta=1}^d \sum_{\alpha \in I} \sum_{\beta \neq \alpha} \langle \sigma^\theta_{\alpha,\beta}, (D^2_{\alpha,\beta}\psi)\sigma^\theta_{\alpha,\beta} \rangle\,,$$

[43] That is $\{q_i\}$ has no accumulations points in $\mathbb{R}^3$.

[44] In fact, it must satisfy some extra properties of a more technical nature (see [54]for details).

[45] The Liouville operator is nothing else than the generator of the flow associated to the Hamiltonian vector field.

[46] In the following we assume also that they satisfy some extra technical conditions, see [26] for details.

where $D^2_{\alpha,\beta}\psi$ is the matrix of second derivatives obtained by applying $D_{\alpha,\beta} = \partial/\partial_{p_\alpha} - \partial/\partial_{p_\beta}$ twice to ψ.

Without entering in further technical details let us sum up the situation by saying that the stochastic interaction is a diffusion conserving the total energy and momentum, thus it can really be thought of as the result of some complex internal motion of the constituents of the particles.

In this situation the concept of ergodicity that proves the most fruitful is the above mentioned classification of the invariant measures in the class of measures of finite specific entropy.

The strongest results in the above direction can be found in [26], [54] and refer to a generalization of the model discussed above: the stochastic interaction can be arbitrarily weak (ε arbitrarily small) and of short range.[47] Let us call $\rho(\omega) = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}|\omega_\Lambda|$ the density of particles in the configuration $\omega = (q,p)$.

Theorem 3.1 ([54], [26])**.** *Suppose that P is a translational invariant stationary measure with finite specific entropy, and let $\rho_c := 3/(4\pi R_1^3)$. If $P(\{\omega \mid \rho(\omega) > \rho'\}) = 1$ for some $\rho' > \rho_c$, then P is a convex combination of Gibbs measures.*

Certainly the value of ρ_c in the previous Theorem it is not optimal, yet when $\rho \to 0$ the system gets closer and closer to the free gas since the particle are, in average, more and more distant and consequently tend to not interact. It is therefore natural that some complication arises for small ρ. If the problem is a real one or just an artifact of the method of proof it remains to be seen.

To conclude these brief comments it is worth mentioning that, in general, no system is really isolated. One can then wonder about the influence of the external world on the system under consideration. A typical way to do this is by introducing stochastic boundary conditions. This is a rather vast subject that it is not reasonable to review it here, let us just mention [27] where it is shown that a gas of interacting particles with stochastic boundary conditions has indeed a stationary measure, [57] where the Lorenz gas with stochastic boundary conditions is investigated and [23] where a stationary measure is obtained for an an-harmonic chain connected to two different stochastic boundary conditions that simulate two heat bath at different temperatures. This should give a feeling for the fact that, once stochasticity is introduced in the system a priori, much stronger results can be obtained.

The above results can also be seen as a stability result for Gibbs states.[48] In fact, a diffuse point of view is that there is no need of ergodicity to justify statistical mechanics, it suffices to notice that the Gibbs states are the only stable states with respect to small perturbations together with the fact that the functions we are interested in are almost constant (with respect to any Gibbs measure) on the phase space. Such a point of view has certainly its merits but, as we will see in the next lecture, it is not completely clear to which extent the above mentioned stability holds.

4. Fourth Lecture

Let us finally get to quantum mechanics. As already mentioned the amount of results is quite limited. Some of the most relevant are [21], [19], [59], [12], [28] where

[47] The range R_1 of the stochastic interaction must be larger than the range R_0 of the deterministic interaction, also, for simplicity, the deterministic potential is assumed to be repelling.

[48] Since the noise level can be taken arbitrary small.

models similar to the one discussed below are studied; [37] and [35] which deals with the radiative decay in the so called spin-boson model; [62] that discusses general features of quantum mixing; [50], [63] where the ergodic properties of quantum systems subject to the Maxwell–Boltzmann statistics are investigated. Such a list is far from exhaustive but, together with the references in the mention articles, should give some idea of the state of the matter.

The above results are certainly important but not completely satisfactory from the point of view of the previous lecture since they limit their study to the KMS states.[49]

In the following we will discuss an attempt, carried out in [24], to go beyond the KMS states, the price to pay is the necessity to restrict our investigation to a model very close to a free gas model. Before going to the concrete example few general remarks are called for.

4.1. **On Quantum Ergodicity.** At this point it may be helpful to make precise what I mean by the *ergodic properties of a quantum dynamical systems*, since such concepts may be less familiar than their classical counterparts.

We consider the couple $(\mathfrak{A}, \alpha_t)$ where $\mathfrak{A}$ is a C^*-algebra and $t \to \alpha_t$ is a representation of the additive group of real numbers into the group of the automorphisms of the C^*-algebra $\mathfrak{A}$, that is a dynamical system.[50]

We say that a representation π of $\mathfrak{A}$ on a separable Hilbert space gives rise to a *continuous dynamical system* if the maps

$$t \in \mathbb{R} \to \pi(\alpha_t A)$$

are all continuous in the weak (equiv. strong, strong*) operator topology.

As it is widely known, there are very simple examples satisfying the above picture, e.g. the dynamical system associated to a free particle. Moreover, if we deal with a group of Bogoliubov transformations $t \to T_t$ of a CCR C^*-algebra built on a symplectic space (H, σ), the Gelfand-Naimark-Segal (GNS for short) representation π_ω relative to any state ω gives rise to a continuous dynamical system according to the above definition, provided that both functions

$$\begin{aligned} t \in \mathbb{R} \to & \sigma(u, T_t v) \\ t \in \mathbb{R} \to & \omega(W(u + T_t v)) \end{aligned}$$

are continuous for fixed $u, v \in H$, [16].

It is straightforward to show that, if ω is an invariant state satisfying the above conditions, then the time evolution is implemented on the GNS Hilbert space $\mathcal{H}_\omega$ by an one-parameter weakly continuous unitary group. Thus, the GNS triple relative to ω leads to a continuous covariant representation for the system $(\mathfrak{A}, \alpha_t, \omega)$.

In this paper, a triple $(\mathfrak{A}, \alpha_t, \omega)$ made of the dynamical system $(\mathfrak{A}, \alpha_t)$ together with an invariant state ω such that the GNS representation π_ω gives rise to a continuous dynamical systems in the above sense, is said to be a *Quantum Dynamical System* (QDS for short).[51]

Concerning ergodic properties of a QDS $(\mathfrak{A}, \alpha_t, \omega)$ with ω invariant, we introduce the following definitions (see e.g. [33, 41]):

[49] The KMS states are the analogous, in quantum mechanics, of the classical Gibbs measure, see [16] for details.

[50] This is nothing else that the definition of classical dynamical system lifted at the level of the algebra of the observable and extended to non commutative algebras.

[51] This is obviously the generalization to the quantum situation of Definition 1.

(i) $(\mathfrak{A}, \alpha_t)$ is *asymptotically Abelian* if

$$\lim_{t\to\infty} \|[\alpha_t A, B]\| = 0$$

for every $A, B \in \mathfrak{A}$.

(ii) $(\mathfrak{A}, \alpha_t, \omega)$ is *ergodic* if ω is the only invariant state for α_t in $\pi_\omega(\mathfrak{A})''_*$.

(iii) $(\mathfrak{A}, \alpha_t, \omega)$ is *weakly mixing* if

$$m_t(\omega(A(\alpha_t C)B) - \omega(AB)\omega(C)) = 0 \tag{4.1}$$

for every $A, B, C \in \mathfrak{A}$ and m_t is any invariant mean on $\mathbb{R}$.

(iv) $(\mathfrak{A}, \alpha_t, \omega)$ is *mixing* if

$$\lim_{t\to\infty} \omega(A(\alpha_t C)B) = \omega(AB)\omega(C) \tag{4.2}$$

for every $A, B, C \in \mathfrak{A}$.

We note that a state ω enjoying properties (iii) or (iv) is referred, in the current literature, as a *weakly clustering* or *strongly clustering* state respectively, see e.g. [16, 17, 33, 41, 62]. Yet, in analogy with the classical case, we prefer to retain our terminology since properties (iii) and (iv) are the natural generalizations to quantum (i.e. non commutative) cases of analogous properties considered in commutative cases, see e.g. [4, 31].

Noticing that we have the chain of implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) for the above properties ([41], Theorem 2.1), we focus our attention only on properties (i) and (iv).

Finally, we remark that, for asymptotically Abelian QDS $(\mathfrak{A}, \alpha_t, \omega)$, property (iv) is equivalent to

$$\lim_{t\to\infty} \omega(A\alpha_t B) = \omega(A)\omega(B) \tag{4.3}$$

for every $A, B \in \mathfrak{A}$ (two-point clustering).[52]

A tool often used in C^* or W^*-dynamical systems in order to investigate asymptotic properties is the following strengthening of (i):

(v) $(\mathfrak{A},\, \alpha_t)$ is L^1*-asymptotically Abelian* if there exists a norm-dense $*$-subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ such that

$$\int_0^\infty \|[\alpha_t A,\, B]\| dt < +\infty$$

for every $A, B \in \mathfrak{A}_0$.

The results of the following section rest on the proof of a weaker version of L^1 asymptotic Abelianess.

Notice that the previous definitions fall under the criticisms of section 3.3. At the moment it is not completely clear (at least to me) how to extend the considerations of section 3.3 to the quantum setting. Nevertheless, we learned there that, at least, all Gibbs measures (including the one corresponding to Hamiltonians non related to the one defining the dynamics) should be considered. In the following we will assume an even more restrictive point of view: since the model we will discuss first is an harmonic chain to which corresponds a quadratic Hamiltonian, we will consider only states determined by quadratic Hamiltonians (that is, *quasi-free* states) and their perturbations.

It is now time discuss a concrete example.

[52]For KMS states of C^*-dynamical or W^*-dynamical systems, the equivalence between (4.2) and (4.3) holds independently of asymptotic abelianess.

4.2. **A weakly an-harmonic quantum chain.** We begin our analysis considering the Hamiltonian

$$\begin{aligned} H(q,p) =& \frac{1}{2m}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\left(\frac{1}{M}-\frac{1}{m}\right)p_0^2 + \frac{K_0}{2}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 \\ &+ \sum_{i\in\mathbb{Z}}\frac{\kappa}{2}q_i^2 + V_*(q_0)\,. \end{aligned}$$

This corresponds to a harmonic chain with one heavier particle subject to a possibly non quadratic potential.

For simplicity we choose the unity of measure such that $m = 1$, $4K_0 = 1$ and $\hbar = 1$. Thus we obtain

$$\begin{aligned} H(q,p) =& \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\left(\frac{1}{M}-1\right)p_0^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 \\ &+ \sum_{i\in\mathbb{Z}}\frac{\kappa}{2}q_i^2 + V_*(q_0)\,. \end{aligned} \tag{4.4}$$

As a preliminary step, let us start by considering the special case in which $V_*(q) = \frac{K}{2}q^2$, $K > 0$, which yields the quadratic Hamiltonian

$$\begin{aligned} H(q,p) =& \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\left(\frac{1}{M}-1\right)p_0^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 \\ &+ \sum_{i\in\mathbb{Z}}\frac{\kappa}{2}q_i^2 + \frac{K}{2}q_0^2\,. \end{aligned} \tag{4.5}$$

Clearly H represents an infinite harmonic chain with a particle of different mass subject to an external potential. One can consider both the case $\kappa > 0$ and the case $\kappa = 0$, when an infrared divergence is present; for simplicity let us restrict the discussion to the $\kappa > 0$ case.

In order to study the evolution of such a (quantum) system, it is natural to consider the real vector space $L^2_{\mathbb{R}}(\mathbb{Z})^2 := L^2_{\mathbb{R}}(\mathbb{Z}) \oplus L^2_{\mathbb{R}}(\mathbb{Z})$ of the doubly infinite sequences that are square summable and the symplectic space $(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)$ with symplectic form

$$\sigma(v,\, w) = \frac{1}{2}\sum_{i\in\mathbb{Z}} v_i^1 w_i^2 - v_i^2 w_i^1,$$

for each $v = (v^1,\, v^2)$, $w = (w^1,\, w^2) \in L^2_{\mathbb{R}}(\mathbb{Z})^2$.

We introduce the block operator $A_\mu : L^2_{\mathbb{R}}(\mathbb{Z})^2 \to L^2_{\mathbb{R}}(\mathbb{Z})^2$

$$A_\mu = \begin{pmatrix} 0 & I - a\mathcal{P} \\ \frac{1}{4}\Delta + \mu I - K\mathcal{P} & 0 \end{pmatrix}$$

where $\mu = -\kappa$,[53] $a = 1 - \frac{1}{M}$ and $\mathcal{P}$, Δ are operators from $L^2_{\mathbb{R}}(\mathbb{Z})$ to $L^2_{\mathbb{R}}(\mathbb{Z})$ defined by $(\mathcal{P}v)_i = \delta_{i0}v_0$ and $(\Delta v)_i = v_{i+1} + v_{i-1} - 2v_i$, respectively.[54]

[53] This notation is convenient since the parameter μ can be interpreted as a kind of "chemical potential" in analogy with [17], Section 5.2.

[54] The Hamilton equations associated to (4.5) read

$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = A_\mu \begin{pmatrix} q \\ p \end{pmatrix}.$$

It is well known (see e.g.[1, 2, 3, 17, 40, 56, 61, 73]) that, in a quantum system, the observables are described by a CCR algebra over (appropriate subspace of) $L^2_{\mathbb{R}}(\mathbb{Z})^2$. More precisely, to each symplectic subspace $H \subset L^2_{\mathbb{R}}(\mathbb{Z})^2$, it is uniquely associated a C^*-algebra $\overline{\mathfrak{A}(H,\sigma)}$.[55]

The algebras of observables that will be relevant in the sequel are determined by the Banach spaces $W^n(\mathbb{Z})$ made of sequences $v \equiv \{v_k\}$ such that

$$\|v\|_{W^n} := \left[\sum_{k\in\mathbb{Z}} (|k|^n + 1)|v_k|\right] < +\infty$$

(of course $W^0 = L^1$).

It is immediate to verify that A^*_μ generate a one-parameter group of Bogoliubov automorphisms (simply the adjoint of the classical evolution)

$$T^\mu_t v := e^{tA^*_\mu} v, \quad t \in \mathbb{R} \tag{4.6}$$

on each of the symplectic spaces $(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)$ and $(W^n_{\mathbb{R}}(\mathbb{Z})^2, \sigma)$.

Thus, the dynamics relative to the Hamiltonian (4.5) is uniquely determined on all C^*-algebras $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ and $\overline{\mathfrak{A}(W^n_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ by the following action on the generators:

$$\alpha^{\mu,0}_t W(v) := W(T^\mu_t v), \quad t \in \mathbb{R}. \tag{4.7}$$

It turns out that the C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ is asymptotically Abelian w.r.t. the dynamics $\alpha^{\mu,0}_t$, $\mu < 0$, whereas the C^*-algebra $\overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ is asymptotically Abelian w.r.t. the dynamics $\alpha^{0,0}_t$.

The possibility to consider various spaces for the observables resembles quite a bit the classical situation where there was the possibility to choose different Banach spaces and the difference did matter. Yet, contrary to the classical case, here the space of observables is ridiculously small. Naïvely it corresponds to little more than finite combination of operators of the type $e^{i\sum_{k\in\mathbb{Z}} v^1_k q_k + v^2_k p_k}$ where $\{q_k, p_k\}$ are the position a momentum of the k-particle. Thus for finitely many particles (when all the spaces we are considering coincide) the observables under consideration correspond, classically, to something like the uniform closure of the trigonometric polynomial. This is a quite typical situation for CCR algebras: they tend to be rather small. Of course, once the state is specified one could consider the associated Von Neumann algebra, this classically would correspond to the $L^\infty(\mu)$ functions, where μ is the measure defining the state of the system. Yet, the discussion on the classical case has clarified that this space is most likely too big. It is not surprising then that it is not known if the results discussed in the following can be extended to the Von Neumann algebra; indeed more likely they are false for such a large set of observables. The situation is thus unsatisfactory and much more needs to be understood. This will emerge more clearly when we will discuss the non linear dynamics. The above CCR is so small that it is not even an invariant algebra for the non linear dynamics. We will solve the problem in a minimalist way: we will enlarge the algebra the minimum necessary to be able to define the dynamics,

[55]Given a symplectic subspace $H \subset L^2_{\mathbb{R}}(\mathbb{Z})^2$, the associated CCR C^*-algebra $\mathfrak{A}(H,\sigma)$ is generated by the Weyl operators $\{W(v)\}_{v\in H}$ with the relations $W(0) = I$, $W(-v) = W(v)^*$, and $W(v)W(w) = e^{-i\sigma(v,w)}W(v+w)$, [40]. Notice that the C^* norm is uniquely determined by the algebraic structure [17], Theorem 5.2.8. Then $\overline{\mathfrak{A}(H,\sigma)}$ is the norm closure of $\mathfrak{A}(H,\sigma)$.

essentially our final algebra will be vaguely reminiscent, in the classical case, of the space of functions analytic in a strip.

The last element of the model is the class of states we intend to consider. As anticipated at the end of the section 4.1 we introduce a wide class of invariant quasi-free states ω for our chain and limit the discussion to their statistical properties.

Quasi-free states are the quantum analogous of a Gaussian probability measure. As in the Gaussian case they are completely determined by a quadratic form B on the space V on which the CCR algebra $\mathfrak{A}(V,\sigma)$ is defined. More precisely the quasi free state ω is determined by the quadratic form B, often called the *two point function*, by the formula (see e.g. [1, 2, 17, 61, 73])

$$\omega(W(v)) = e^{-\frac{1}{2}B(v,v)}. \tag{4.8}$$

Clearly, by using the commutation relations of the CCR (see footnote 55), relation (4.8) allows to compute the expectation of all the observables. For the linear quantum system under investigation it turns out that the KMS states are indeed quasi-free states with two point function given by

$$B_F(u,v) = \langle u, F_\beta(A_\mu)v\rangle \tag{4.9}$$

where F_β is the analytic function

$$F_\beta(z) := \frac{\coth(\frac{\beta}{2}\sqrt{-z})}{\sqrt{-z}}.$$

It is easy to check that the two-point function (4.9) gives rise to a KMS state $\omega_{\mu,\beta}$ at the inverse temperature β for the CCR C^*-algebra $\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}$ w.r.t. the dynamics $\alpha_t^{\mu,0}$. More interesting is the existence of many other quasi–free states invariant under the dynamics. They can be viewed as KMS stated of different dynamics that *commute* with our dynamics. The physical interpretation of this phenomena is rather obvious: the harmonic chain is an *integrable system*, accordingly its normal modes evolve independently. Hence, any state that diagonalizes on the normal modes and is invariant under the evolution of each single normal mode will be invariant. Clear there exists an enormous amount of arbitrariness on which statistical weights attribute to the various normal modes.

It is possible to show that the Quantum Dynamical System $(\overline{\mathfrak{A}(L^2_{\mathbb{R}}(\mathbb{Z})^2,\sigma)}, \alpha_t^{\mu,0}, \omega)$ associated to any invariant quasi-free state ω is mixing. Such states, in particular, include the KMS states and ground states.

4.2.1. *The non linear case.* As already mentioned the definition of the dynamics $\alpha_t^{\mu,P}$ associated to the Hamiltonian (4.4) with

$$V_*(x) := \frac{K}{2}x^2 + V(x)$$

is more delicate. The main obstacle is that the operator associated to the potential V is not necessarily an element of the above C^*-algebras.

To be more precise we will consider

$$V(x) = \int_{\mathbb{R}} e^{i\lambda x}\nu(d\lambda)$$

where ν is a bounded complex measure with real Fourier transform. Clearly, the self-adjoint operator P naturally associated to $V(x)$ should be

$$P := \int_{\mathbb{R}} W(\lambda e_{0,1})\nu(d\lambda) \tag{4.10}$$

where $e_{0,1} = (e_0, 0)$ and, as usual, $e_0 = (\ldots, 0, 1, 0, \ldots)$. Unfortunately the above integral has meaning only in any of weak topologies associated to a suitable representation π of $\mathfrak{A} = \overline{\mathfrak{A}(W^2_{\mathbb{R}}(\mathbb{Z})^2, \sigma)}$ (e.g. the weak*-topology of $\pi(\mathfrak{A})''$), not in the norm topology. It is thus necessary to enlarge in a canonical way our C^*-algebras in order to define QDSs in the non linear case. This new algebra of observables will depend on the state ω under consideration. The basic idea to construct it is to add to the CCR all the element of the associated Von Neuman algebra generated by weak integration, namely for each complex measure on $W^2_{\mathbb{R}}(\mathbb{Z})^2$ such that for some $\varepsilon > 0$

$$\int_{W^2_{\mathbb{R}}(\mathbb{Z})^2} e^{\varepsilon \|x\|_{W^2_{\mathbb{R}}(\mathbb{Z})^2}} |\mu|(dx) < \infty$$

we consider the operator

$$\int_{W^2_{\mathbb{R}}(\mathbb{Z})^2} W(x)\mu(dx) \tag{4.11}$$

where the integral is understood with respect to the strong operator topology. The wanted C^*-algebra can then be obtained as the norm closure of the above set of operators. Note that the operator (4.10) associated to the non-quadratic part of the potential is a particular case of (4.11), thus it is quite reasonable that the above algebra is large enough to define the non-linear dynamics while, at the same time, is more or less as small as it can be. We will call such an algebra $\mathfrak{M}_\omega$.

As it is well known [16, 17], the dynamics can then be defined by the equation

$$\alpha^P_t A = \alpha^0_t A + i \int_0^t ds \alpha^P_s [P, \alpha^0_{t-s} A] \tag{4.12}$$

where α^0_t denotes the linear dynamics extended to $\mathfrak{M}_\omega$. In our case P is precisely the non quadratic part of the external potential acting on the 0^{th}-particle.

The main result can then be summarized by the following theorem.

Theorem 4.1. *Suppose that the potential V is sufficiently small.*
Then the following assertions hold.

(i) *To any quasi-free state ω on $\mathfrak{A}$ invariant w.r.t. the linear dynamics $\alpha^{\mu,0}_t$, such that $v \in W^2_{\mathbb{R}}(\mathbb{Z})^2 \to \omega(W(v)) \in \mathbb{C}$ is a Borel function, it corresponds a unique state ω_∞ on $\mathfrak{M}_\omega$ invariant w.r.t. the non linear dynamics $\alpha^{\mu,P}_t$, which can be obtained as the following pointwise limit:*

$$\omega_\infty(A) = \lim_{t\to\pm\infty} \omega(\alpha^{\mu,P}_t A)\,.$$

(ii) *If the QDS $(\mathfrak{A}, \alpha^{\mu,0}_t, \omega)$ is mixing, then the QDS $(\mathfrak{M}_\omega, \alpha^{\mu,P}_t, \omega_\infty)$ is mixing.*

The meaning of *sufficiently small* in the above theorem is specified by requiring that the quantity

$$[\![\nu]\!] := \sup_{n\in\mathbb{N}} \frac{1}{(n-1)!} \int_{\mathbb{R}} |\nu|(d\lambda)|\lambda|^n, \tag{4.13}$$

be sufficiently small. In particular $[\![\nu]\!] \leq \infty$ implies that V is analytic. The exact amount of smallness required it is not so relevant here, it suffices to say that the limit is more or less given by the requirement that the potential remains concave.

The technique used to establish the above result is perturbation theory.

Indeed, a crucial step in the proof is the derivation of the following integral equation

$$f(v) = \omega(W(v)) + \int_0^\infty ds \int_{\mathbb{R}} \nu(d\lambda) f(\lambda e_{0,1} + T_s v) g(v, s, \lambda). \tag{4.14}$$

where f belongs to the class of locally bounded Borel function on the symplectic space and g is a kernel determined by the commutation rule between Weyl operators (see footnote 55).[56] The equation (4.14) has a unique solution given precisely by $f(v) = \omega_\infty(W(v))$. This is proven by the standard trick of iterating the integral equation and proving convergence of the resulting series. The fact that the perturbative series for the dynamics can be controlled uniformly in time is a bit surprising and can hardly be expected to hold in general.

Nevertheless, the above equation can be easily recovered in a more general setting and could be the starting point for a non perturbative study of the approach to equilibrium (see [71] for very recent related results).

In conclusion, we have that non–KMS states survive for the perturbed dynamics. Clearly this contradicts the naïve idea that the KMS states are the only states that survive perturbations. Of course, the partisan of this point of view may rightly argue that the perturbations considered here are unreasonably small and that to have a real argument one should be able to treat, at least, a weakly anharmonic crystal (that is the non-linear perturbation should be possibly very small, but present on each spring). Since such a perturbation in unbounded it does not fall in the previous setting. Nevertheless one should emphasize that also the rigorous results on the stability of the KMS states are not completely satisfactory. The problems being that they require stability is a rather strong sense and with respect to a very wide class of perturbations (see, e.g., [70]).

I think it is then fair to say that the issue it is not yet satisfactorily solved and more work is needed to clarify theses questions.

I would like to emphasize that the speed of mixing in the previous systems is only polynomial (versus the classical examples which often exhibit exponential mixing). This is another indication of a more complex situation. Yet, also classically there are several cases in which the decay is only polynomial (for example intermittent maps or non–Hölder potentials[57]) and they can still be treated with methods similar to the one outlined in these notes.

4.3. **Concluding remarks.** In this lectures I have tried to outline some results and techniques concerning the return to equilibrium in classical and quantum systems. The discussion has emphasized some major differences between the two settings.

[56] To obtain (4.14) take the average of (4.12) with respect to ω and compute.

[57] By intermittent maps are meant maps with a neutral fixed point, for example consider an expanding map of the interval with zero as a fixed point such that, near zero, has the form $x + x^{1+\alpha}$, $\alpha \in (0, 1)$. Such maps have a unique invariant measure absolutely continuous with respect to Lebesgue (although the density is singular at zero) and are mixing (if topologically mixing), but the rate of mixing it is only polynomial, see [55], [75], [34], [14]. For the case of non-Hölder potential see [59], [69], [15].

The most important being that in the quantum framework one is forced to consider immediately infinite systems and therefore there is a lack of simple models. In fact, all the classical theory stars with the study of systems with few degree of freedom (one or two dimensions) and has focused on the phenomena of sensitive dependence from initial conditions (hyperbolicity). In the quantum setting such concepts are lacking (e.g., there is no clear definition of what a quantum Lyapunov exponent could be) or have many possible generalizations (e.g., the concept of entropy) and it is unclear which should be used to produce results analogous to the classical setting. Some interesting attempts to address such questions have been done (notably all the work under the label of *Quantum Chaos*, see [72] for some rigorous results and [36], [30] for an overview of the field). Yet, the situation is still vastly unsatisfactory.

This state of affair reflects on the fact that, although I have made an effort to present the classical theory in a manner as close as possible to the quantum setting, it is not clear how to really generalize the classical ideas. In fact, when I finally presented some results for quantum systems they were essentially based on perturbative theory while the fundamental feature of the classical methods previously introduced is their non perturbative, but rather *dynamical*, character. As far as I can see such *dynamical* arguments are still missing in the quantum setting and their development stand as a challenge for the future.

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