SMOOTH INVARIANT CONE FIELDS: ERRATA CORRIGE TO "SMOOTH ANOSOV FLOWS: CORRELATION SPECTRA AND STABILITY"

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Abstract. The Lemma 7.2 of the paper "Smooth Anosov Flows: Correlation Spectra and Stability" was incorrect. Here we fix the error. As part of the solution we show that it is possible to construct smooth invariant cone fields for Anosov flows.

1. Introduction

Lemma 7.2 of the article "Smooth Anosov Flows: Correlation Spectra and Stability" [2] was stated to hold for all $t \geq 0$ but in fact it is only true for all t greater than some non-zero constant. This affects all the paper since [2] reduces to the study of the transfer operator, associated with the dynamics, on appropriate Banach spaces. The norms defining such Banach spaces are based on a set of *admissible leaves* that are required to be *invariant*, with respect to the dynamics, in the sense of [2, Lemma 7.2]. Since the set of leaves defined in [2] is invariant only after some finite time (contrary to the statement of [2, Lemma 7.2]) no control was obtained on the transfer operator for short time.

To solve the problem it is then necessary to change slightly the definition of the Banach space. This can be done in several ways, a simple solution (used in [1, 3]) is to keep the definition of leaves in [2] and use the dynamics explicitly in the definition of the norm (by taking the new norm to be the sup of the old ones over some time interval). Unfortunately, this is not suited for the task at hand as we are interested also in perturbations of a given dynamics, hence we do not wish to have a norm too closely tied to the dynamic. One could probably fix the latter issue by taking the sup not only on time but also on a neighbourhood of time dependent dynamics. Yet, this would make the definition of the space rather cumbersome. Instead, we choose to redefine the set of admissible leaves by an explicit construction that has also the merit to better elucidate the geometric properties of the dynamics and that could be useful in related problems. Regrettably, in so doing we lose the compact embedding between the Banach spaces [2, Lemma 2.2], which previously held thanks to [4, Lemma 2.1]. The latter does not apply given the new definition of the leaves. Nevertheless, it is easy to recover the needed compactness at the level of resolvent operators. The previously stated results are thereby proved.

To define the new set of leaves we first define an appropriate norm in the tangent space, then a smooth invariant cone field (the existence of which is not obvious since it is known that the invariant splitting is, in general, not better than Hölder. The

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problem being, again, small times). Finally we establish the needed control on the higher derivatives of the leaves. To avoid further mistakes we will do such construction explicitly, also in view of the fact that it is of independent interest. In essence we improve (and extend to the case of flows) an old trick of Mather [7]. The punch line is given by Theorem 5.2 that takes the place of [2, Lemma 7.2].

Once the new set of admissible leaves (and hence the new norms) are defined we will explain exactly which part of the arguments in [2] must be modified to take into account the new definition of the Banach space.

2. The Metric (part I)

Assume that $\mathcal{M}$ is a $\mathcal{C}^\infty$ compact orientable Riemannian manifold and T_t an Anosov flow. Here by Anosov flow we mean that there exists a continuous splitting $T\mathcal{M} = E^s \oplus E^u \oplus E^f$, $\dim(E^f) = 1$, of the tangent space and $C_* \in (0,1], \lambda > 0$ such that

$$\begin{aligned} &\|D_x T_t v\| \geq C_* e^{\lambda t}\|v\| \quad \text{for all } t \geq 0, v \in E^u(x) \\ &\|D_x T_{-t} v\| \geq C_* e^{\lambda t}\|v\| \quad \text{for all } t \geq 0, v \in E^s(x) \\ &V(x) \neq 0 \quad \text{for all } x \in \mathcal{M} \end{aligned} \tag{2.1}$$

where $\|\cdot\|$ is the Riemannian metric and $V \in \mathcal{C}^{r+1}, V(x) \in E^f(x)$, is the vector field generating the flow T_t. Then $T_t \in \mathrm{Diff}^{r+1}(\mathcal{M}, \mathcal{M})$. Also, the above implies that there exists $C_1 > 1$ such that $C_1^{-\frac{1}{2}} \leq \|V(x)\| \leq C_1^{\frac{1}{2}}$ for all $x \in \mathcal{M}$. We start by showing that any Anosov flow satisfies [2, Condition 2].[1]

Lemma 2.1. *For each $\mathcal{C}^{r+1}$ Anosov flow there exist a $\mathcal{C}^r$ Riemannian metric $\|\cdot\|_1$, uniformly equivalent to the original Riemannian metric, and $\sigma \in (0, \lambda)$ such that*

$$\|D_x T_t v\|_1 \geq e^{\sigma t}\|v\|_1 \quad \textit{for all } t \geq 0, v \in E^u(x)$$
$$\|D_x T_{-t} v\|_1 \geq e^{\sigma t}\|v\|_1 \quad \textit{for all } t \geq 0, v \in E^s(x)$$
$$\max\{C_1^{-1}, e^{-\frac{\sigma}{4}t}\}\|v\|_1 \leq \|D_x T_t v\|_1 \leq \min\{C_1, e^{\frac{\sigma}{4}t}\}\|v\|_1 \quad \textit{for all } t \in \mathbb{R}_+, v \in E^f(x).$$

Proof. The construction of the norm is based on two parameters $L > A > 0$, such that $C_* e^{\frac{\lambda}{2}A} \geq 1$, that will be chosen along the proof. We define the new norm

$$\langle v, w\rangle_1 = L^{-1}\int_0^L \langle DT_s v, DT_s w\rangle ds\,; \qquad \|v\|_1 = \sqrt{\langle v, v\rangle_1}.$$

Consider first the case $v \in E^u$. We divide $\mathbb{R}_+$ into short times $t \leq A$ and large times $t > A$. Let us start analyzing the short times.

$$\begin{aligned} L\|DT_t v\|_1^2 &= \int_t^{L+t} \|DT_s v\|^2 ds \\ &= L\|v\|_1^2 + \int_0^t \|DT_{L+s} v\|^2 ds - \int_0^t \|DT_s v\|^2 ds \\ &\geq L\|v\|_1^2 + (1 - C_*^{-2} e^{-2\lambda L})\int_0^t \|DT_{L+s} v\|^2 ds \\ &\geq L\|v\|_1^2 + \frac{(1 - C_*^{-2} e^{-2\lambda L})t}{(1-a)L}\int_{aL}^L \|DT_{L+\frac{\tau - aL}{(1-a)L}t} v\|^2 d\tau \end{aligned} \tag{2.2}$$

[1] This was claimed in [2, Footnote 1]). Since this issue is more crucial than anticipated, and some extra properties are need, we prefer to give a detailed proof of this fact.

where, in the last line, we performed the change of variables $s = \frac{\tau - aL}{(1-a)L}t$ and we have introduced $a \in (\frac{1}{2}, 1)$ to be chosen shortly. Setting $\beta = \frac{C_*^2 e^{2\lambda(1-a)L}(1-a)}{a+(1-a)C_*^2 e^{2\lambda(1-a)L}}$,

$$\begin{aligned}\int_{aL}^{L} \|DT_s v\|^2 ds &= \beta \int_{aL}^{L} \|DT_s v\|^2 ds + \frac{(1-a)(1-\beta)}{a} \int_0^{aL} \|DT_{aL+\frac{1-2a}{a}s+s} v\|^2 ds \\ &\geq \beta \int_0^L \|DT_s v\|^2 ds.\end{aligned}$$

Using (2.2) together with the above estimate and choosing $a = 1 - AL^{-1}$, we have[2]

$$\begin{aligned}L\|DT_t v\|_1^2 &\geq L\|v\|_1^2 + \frac{(1 - C_*^{-2}e^{-2\lambda L})t}{(1-a)L} C_*^2 e^{2\lambda t} \int_{aL}^{L} \|DT_\tau v\|^2 d\tau \\ &\geq L\left[1 + \beta \frac{(1 - C_*^{-2}e^{-2\lambda L})t}{A} C_*^2 e^{2\lambda t}\right] \|v\|_1^2.\end{aligned}$$

Note that there exists $K > 0$ such that $\beta(1 - C_*^{-2}e^{-2\lambda L}) \geq \frac{1}{2}$, provided $A \geq K \ln L$ and $L \geq K$ (which implies that we must choose $L \geq 2K \ln L$). Thus

$$\|DT_t v\|_1 \geq e^{\frac{C_*^2}{4A}t} \|v\|_1.$$

On the other hand, if $t > A$, then

$$\|DT_t v\|_1 \geq C_* e^{\lambda t} \|v\|_1 \geq e^{\frac{\lambda}{2}t} \|v\|_1. \tag{2.3}$$

Next, we consider $v \in E^s$. We can argue in complete analogy with the above computation apart from the fact of using $[0, (1-a)L]$ as reference set instead than $[aL, L]$. In so doing we obtain again

$$\|DT_{-t} v\|_1 \geq e^{\sigma_A t} \|v\|_1,$$

for $\sigma_A = \min\{\frac{\lambda}{2}, \frac{C_*^2}{4A}\}$. Finally, if $v \in E^f(x)$, then $v = \alpha V(x)$. Since $DT_t V = V \circ T_t$ we have, for all $t \in \mathbb{R}$,

$$C_1^{-1} \|v\|_1 \leq \|DT_t v\|_1 \leq C_1 \|v\|_1.$$

While, for small times,

$$\begin{aligned}L\|DT_t v\|_1^2 &= \int_t^{L+t} \|DT_s v\|^2 ds \\ &= L\|v\|_1^2 + \alpha^2 \int_0^t \|V \circ T_{L+s}\|^2 ds - \alpha^2 \int_0^t \|V \circ T_s\|^2 ds \\ &\geq L[1 - tL^{-1}(C_1^2 - C_1^{-2})] \|v\|_1^2 \\ &\geq L e^{-\frac{\sigma_A}{4}t} \|v\|_1^2,\end{aligned}$$

where we have chosen $A = K \ln L$ and L large enough. The other inequality being more of the same. □

Remark 2.2. *What we have done is to find a norm for which (2.1) holds with $C_* = 1$ (with a different $\lambda > 0$). This is very similar to what is proven in [7] for the case of Anosov diffeomorphisms but here there are two (closely related) differences: 1) the metric $\langle \cdot, \cdot \rangle_1$ is $\mathcal{C}^r$ rather than just Hölder as in [7]; 2) contrary to [7] the new distributions are not orthogonal in the new norms. The latter is annoying but*

[2] Note that, for $\tau \in [aL, L]$ and $t \leq A$, $L + \frac{\tau - AL}{(1-a)L}t - \tau \geq L + \frac{L(t-A) - (L-A)t}{A} = t$.

inevitable if one wants a smooth norm. In the following sections we will do the next best thing: we will modify the metric so that the invariant distributions are almost orthogonal.

Remark 2.3. *Note that, in the above proof, σ cannot be taken arbitrarily close to λ, contrary to [7]. This is the price of the naïvity of our construction and the requirement that the metric be smooth. Yet, we will use the above estimate only for short times, for long times equations (2.2) are sufficient and yield optimal bounds.*

Before proceeding further let us notice an easy corollary of Lemma 2.1. Given $v \in T_x\mathcal{M}$ there exists a unique decomposition $v = v^u + v^s + v^f$, where $v^u \in E^u, v^s \in E^s, v^f \in E^f$. For each $x \in \mathcal{M}$ consider the cone

$$C_\rho(x) = \{v \in T_x\mathcal{M} \ : \ \|v^u\|_1 + \|v^f\|_1 \leq \rho\|v^s\|_1\},$$

and its complement C^c_ρ (i.e. $C_\rho \cap C^c_\rho = \{0\}$ and $C_\rho \cup C^c_\rho = T\mathcal{M}$).

Corollary 2.4. *There exists $\rho_0 \in (0,1)$ such that for each $\rho_1 \in (0,\rho_0)$ and $t > 0$*

$$\begin{aligned}
&D_xT_{-t}C_{\rho_1}(x) \subset C_{\rho_1 e^{-\frac{\sigma}{2}t}}(T_{-t}x),\\
&D_xT_tC^c_{\rho_1}(x) \subset C^c_{\rho_1 e^{-\frac{\sigma}{2}t}}(T_tx),\\
&\|D_xT_{-t}v\|_1 \geq e^{\frac{3}{4}\sigma t}\|v\|_1 \quad \forall v \in C_{\rho_1}(x),\\
&\|D_xT_tv\|_1 \geq e^{-\frac{1}{2}\sigma t}\|v\|_1 \quad \forall v \in C^c_{\rho_1^{-1}}(x).
\end{aligned}$$

Proof. As usual we have to worry only about short times. The first two lines follow directly from Lemma 2.1. To verify the third let $v \in C_\rho(x)$, then

$$\begin{aligned}
\|D_xT_{-t}v\|_1^2 =&\|D_xT_{-t}v^s\|_1^2 + \|D_xT_{-t}v^u\|_1^2 + \|D_xT_{-t}v^f\|_1^2 + 2\langle D_xT_{-t}v^s, D_xT_{-t}v^u\rangle_1\\
&+ 2\langle D_xT_{-t}v^s, D_xT_{-t}v^f\rangle_1 + 2\langle D_xT_{-t}v^u, D_xT_{-t}v^f\rangle_1.
\end{aligned}$$

Next, notice that, by the smoothness of the flow and the metric, for each v, w

$$|\langle DT_{-t}v, DT_{-t}w\rangle_1 - \langle v, w\rangle_1| \leq Ct\|v\|_1\|w\|_1.$$

Thus, by Lemma 2.1,

$$\begin{aligned}
\|D_xT_{-t}v\|_1^2 \geq&\|v\|_1^2 + (e^{2\sigma t}-1)\|v^s\|_1^2 - (e^{\sigma t}-1)\left\{\|v^u\|_1^2 + \|v^f\|_1^2\right\}\\
&- Ct\left\{(\|v^s\|_1\|v^u\|_1 + \|v^s\|_1\|v^f\|_1 + \|v^u\|_1\|v^f\|_1)\right\}\\
\geq&\|v\|_1^2 + \left\{e^{2\sigma t} - 1 - [\sigma - C\rho]\,t\right\}\|v^s\|_1^2\\
\geq&\left[1 + \frac{e^{2\sigma t} - 1 - [\sigma - C\rho]\,t}{1+\rho^2}\right]\|v\|_1^2 \geq e^{\frac{3}{4}\sigma t}\|v\|_1^2
\end{aligned}$$

provided ρ_0 is chosen small enough. The last inequality is proven similarly using also the last inequality of Lemma 2.1 since now the flow direction is included inside the cone we are interested in. □

3. Approximate Invariant Foliations

The basic obstacle to overcome is the low regularity of the invariant distributions E^s, E^u. Yet, since the distributions are invariant, i.e. $DT_tE^s(x) = E^s(T_tx)$, they are $\mathcal{C}^r$ when restricted to the flow direction.[3] To take care of the other directions we

[3] I.e., for each $x \in \mathcal{M}$, let $E(t) = E^s(T_tx)$. Then $E \in \mathcal{C}^r(\mathbb{R}, \mathcal{G}^s)$ where $\mathcal{G}^s$ is the Grassmannian of the d_s dimensional subspaces of $T\mathcal{M}$.

can smoothen the distributions obtaining $\hat{E}^s, \hat{E}^u \in \mathcal{C}^r$ ϱ-close, in the $\mathcal{C}^0$ norm, to the invariant ones, $\varrho < \frac{\rho_1}{2}$. In particular, the distributions and their images belong to the cones to which Corollary 2.4 applies. The problem with such a naïve approach is that, given ϱ, the $\mathcal{C}^r$ norm of the approximate distributions is of order $C\varrho^{-\varpi r}$, for some $\varpi > 0$. In particular, the distribution can be "far from invariant" for small times. This is unavoidable, but it is unpleasant that to gain some smoothness in all directions we have lost the original bound on the regularity along the flow direction. To fix the problem we consider the new foliations $\mathcal{E}^s(x) = DT_{-N}\hat{E}^s(T_N x)$, $\mathcal{E}^u(x) = DT_N\hat{E}^u(T_{-N}x)$, for some N large enough. Obviously $\mathcal{E}^s, \mathcal{E}^u \in \mathcal{C}^r$ but their norm will be bounded by some $C_N^r\varrho^{-\varpi r}$.

To point out the usefulness of the above construction we need a bit of notation. Let us restrict ourselves to the unstable foliation, the treatment of the stable being equal. For each $x \in \mathcal{M}$ we can represent a d_u dimensional subspace $E \subset T_x\mathcal{M}$ close to $E^u(x)$ as an operator $U : E^u \to E^s \oplus E^f$, i.e. $E = \{v + Uv \ : \ v \in E^u\}$. Also we use T_t^*U for the action of the dynamics on such operators, i.e. T_t^*U is the operator that represents the space DT_tE. Let $\mathcal{U}(x)$ be the operator associated to $\mathcal{E}^u(x)$. The next Lemma shows that we have recovered control along the flow direction.

Lemma 3.1. *For each $\varepsilon > 0$ there exist $N_\varepsilon > 0$ such that, for all $N \geq N_\varepsilon \ln \varrho^{-1}$, $x \in \mathcal{M}$, and $t > 0$*

$$\|T_t^*\mathcal{U}(x) - \mathcal{U}(T_t x)\| \leq \varepsilon t.$$

Proof. Note that, by the invariance of E^u, $T_t^*0 = 0$. Moreover by standard hyperbolicity estimates it follows that there exists $N_* > 0$ such that $T_{N_*}^*$ is a contraction by more than $\frac{1}{2}$. By definition there exists $\widehat{U}(x)$, $\|\widehat{U}\| \leq \varrho$, such that $\mathcal{U}(x) = T_N^*\widehat{U}(T_{-N}x)$. Thus

$$\begin{aligned}\|T_t^*\mathcal{U}(x) - \mathcal{U}\circ T_t(x)\| &= \|T_{t+N}^*\widehat{U}(T_{-N}x) - T_N^*\widehat{U}(T_{t-N}x)\| \\ &= \|T_N^*(T_t^*\widehat{U}(T_{-N}x)) - T_N^*\widehat{U}(T_{t-N}x)\| \\ &\leq C2^{-\frac{N}{N_*}}\|T_t^*\widehat{U}(T_{-N}x)) - \widehat{U}(T_{t-N}x)\| \leq C2^{-\frac{N}{N_*}}\varrho^{-\varpi}t.\end{aligned}$$

The Lemma follows by choosing N_ε large enough. □

From now on we fix some $\varepsilon \ll \sigma$ and all the other parameters so that the above Lemmata hold true.

Given $v \in T\mathcal{M}$ we consider the decomposition $v = \tilde{v}^s + \tilde{v}^u + \tilde{v}^f$ determined by the splitting $T\mathcal{M} = \mathcal{E}^s \oplus \mathcal{E}^u \oplus E^f$. This induces projectors Π_s, Π_u, Π_f such that $\tilde{v}^s = \Pi_s v$, $\tilde{v}^u = \Pi_u v$ and $\tilde{v}^f = \Pi_f v$. Then the above Lemma can be restated as saying that

$$\begin{aligned}\|(\mathbb{1} - \Pi_s)DT_{-t}\Pi_s\|_1 &\leq C\varepsilon t \\ \|\Pi_s DT_t(\mathbb{1} - \Pi_s)\|_1 &\leq C\varepsilon t\end{aligned} \tag{3.1}$$

while, obviously, $(\mathbb{1} - \Pi_f)DT_t\Pi_f = 0$ and $\Pi_s + \Pi_u + \Pi_f = \mathbb{1}$.

4. The metric (part II)

We are finally in the position to define the required new Riemannian metric

$$\langle v, w\rangle_2 = \langle \tilde{v}^s, \tilde{w}^s\rangle_1 + \langle \tilde{v}^u, \tilde{w}^u\rangle_1 + \langle \tilde{v}^f, \tilde{w}^f\rangle_1,$$

which makes the splitting $\mathcal{E}^s \oplus \mathcal{E}^u \oplus E^f$ orthogonal. Also we define the cone

$$C^*_\rho(x) = \{v \in T_x\mathcal{M} \ : \ \|\tilde v^u + \tilde v^f\|_2 \le \rho \|\tilde v^s\|_2\}.$$

Note that the new metric is also $\mathcal{C}^r$, possibly with a quite large derivative, and hence the cone varies smoothly.

In [2] we are interested in the set of vector fields

$$\mathcal{X}_{\eta_0} = \{X_\eta \ : \ X_\eta = X + \eta X_1, \|X_1\|_{\mathcal{C}^{r+1}} \le 1, |\eta| \le \eta_0\}$$

and we will call $\mathcal{T}_\eta$ the set of flows generated by the vector fields in $\mathcal{X}_\eta$. By the smooth dependency on the vector fields and the initial conditions, for each $\tau > 0$ there exists $C_1, \eta_0 > 0$ such that, for all $T_{\eta,t} \in \mathcal{T}_{\eta_0}$ and $t \le \tau$,

$$\begin{aligned} &\|\mathbb{1} - DT_{\eta,-t}\|_{\mathcal{C}^r} \le C_1 t, \\ &\|DT_{-t} - DT_{\eta,-t}\|_{\mathcal{C}^r} \le C_1 t|\eta|. \end{aligned} \tag{4.1}$$

Lemma 4.1. *There exists $\eta_0, \varsigma, \rho > 0$ such that, for all $T_{\eta,t} \in \mathcal{T}_{\eta_0}$, $x \in \mathcal{M}$ and $t \in \mathbb{R}_+$*

$$\begin{aligned} &\|D_x T_{\eta,-t} v\|_2 \ge e^{\varsigma t}\|v\|_2 \quad \textit{for all } v \in C^*_\rho \\ &\|D_x T_{\eta,t} v\|_2 \ge e^{\varsigma t}\|v\|_2 \quad \textit{for all } v \in \mathcal{E}^u \\ &D_x T_{\eta,-t} C^*_\rho(x) \subset C^*_{\max\{\frac{\rho}{2}, e^{-\frac{\varsigma}{4}t}\rho\}}(T_{\eta,-t}x). \end{aligned}$$

Proof. Remark that, due to the uniform transversality of the invariant distributions, the norms $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_2$ are all uniformly equivalent, moreover the projectors Π_s, Π_u, Π_f are self-adjoint in the metric $\langle\cdot,\cdot\rangle_2$, hence they contract the associated norm. We start by discussing $T_t = T_{0,t}$. Using Corollary 2.4 it is easy to check that there exists $t_* > 0$, independent on $\rho \in (0, \rho_0)$ and ε, such that the Lemma holds trivially for $t \ge t_*$. Let us worry about shorter times.

We choose $\rho \in (0, \rho_0/2)$ large enough so that the stable invariant distributions belong to the cone $C^*_{\rho/2}$, yet $DT_{t_*} C^*_\rho \subset C_{\rho_0/2}$. Let $v \in \mathcal{E}^s$, then $\Pi_s v = v$ and, by (3.1) and Corollary 2.4,

$$\begin{aligned} \|DT_{-t}v\|_2^2 =& \|\Pi_s DT_{-t}\Pi_s v\|_1^2 + \|\Pi_u DT_{-t}\Pi_s v\|_1^2 + \|\Pi_f DT_{-t}\Pi_s v\|_1^2 \\ \ge& \|DT_{-t}v\|_1^2 - C\varepsilon t\|v\|_2^2 \ge e^{\sigma t}(1 - e^{-\sigma t}C\varepsilon t)\|v\|_2^2 \ge e^{2\varsigma t}\|v\|_2^2 \end{aligned} \tag{4.2}$$

for some appropriate ς. By similar computations we can prove the second inequality of the Lemma. Next, for $v \in C^*_\rho$, let $\tilde v^s_t = \Pi_s DT_{-t}v$, $\tilde v^u_t = \Pi_u DT_{-t}v$ and $\tilde v^f_t = \Pi_f DT_{-t}v$, then, remembering (3.1),[4]

$$\begin{aligned} \|\tilde v^u_t + \tilde v^f_t\|_2 &= \|(\mathbb{1} - \Pi_s)DT_{-t}v\|_2 \le \|DT_{-t}(\mathbb{1} - \Pi_s)v\|_2 + C\varepsilon t\|v\|_2 \\ &\le e^{\frac{\varsigma}{2}t}\|(\mathbb{1} - \Pi_s)v\|_2 + C\varepsilon t\|v\|_2 \le e^{-\frac{\varsigma}{2}t}\rho\|DT_{-t}\Pi_s v\|_2 + C\varepsilon t\|v\|_2 \\ &\le (e^{-\frac{\varsigma}{2}t}\rho + C\varepsilon t)\|\tilde v^s_t\|_2 \end{aligned}$$

which proves the third inequality of the Lemma provided $\rho \ge C\varsigma^{-1}\varepsilon$. The first inequality is proven similarly.

To treat the case $\eta \ne 0$, note again that it suffices to prove the result for small times. By (4.1), (4.2)

$$\|DT_{\eta,-t}v\|_2^2 \ge (e^{\sigma t} - C\varepsilon t - C\eta t)\|v\|_2^2 \ge e^{2\varsigma t}\|v\|_2^2,$$

[4] In the first inequality of the second line we use the fact that $DT_{-t}(\mathbb{1} - \Pi_s)v \in C^*_\rho$, the second inequality of the statement and Corollary 2.4.

, provided η_0 is chosen small enough, and the same for the second inequality. The last inequality follows then by the smoothness of the cone field.[5] We can argue similarly for the other inequalities. □

5. The set of admissible leaves

Here we come to the goal of this errata: to restrict the set of allowed manifolds so that it has the wanted invariance also for small times and so that the property persists under perturbations. The first step is to introduce a convenient way to describe a local leaf.

We start by introducing the set $\widetilde{\Sigma}_0$ of the $\mathcal{C}^{r+1}$ d_s-dimensional submanifolds W such that $TW \subset C^*_\rho$. Note that if $W \in \widetilde{\Sigma}_0$, then Lemma 4.1 implies that $T_{-t}W \in \widetilde{\Sigma}_0$ for all $t > 0$.

Fix $B > 2$ large enough. Let Σ_0 be the set of the elements $W \in \widetilde{\Sigma}_0$ such that there exists $W \subset W^+ \in \widetilde{\Sigma}_0$ and $x \in W$ (we will call it its *center*) for which, calling d_W the Riemannian metric induced on W^+ by $\langle \cdot, \cdot \rangle_2$, and setting $W_r = \{z \in W^+ : d_W(x, z) < r\}$ we have $\overline{W}_r \subset W^+$, for each $r < B\delta$, (the closure is meant as a subset of $\mathcal{M}$) and $W_{B\delta} = W^+$, $W_{2\delta} = W$. With the above definitions we have the equivalent of [2, Lemma 7.2].[6]

Lemma 5.1. *There exists $\eta_0 > 0$ such that, for each $T_{\eta,t} \in \mathcal{T}_{\eta_0}$, leaf $W \in \Sigma_0$ and $t \in \mathbb{R}^+$, there exist leaves $W_1, \dots, W_\ell \in \Sigma_0$, whose number ℓ is bounded by a constant depending only on t, such that*

1. *$T_{\eta,-t}(W) \subset \bigcup_{j=1}^{\ell} W_j$.*
2. *$T_{\eta,-t}(W^+) \supset \bigcup_{j=1}^{\ell} W_j^+$.*
3. *There exists a constant C (independent of W and t) such that each point of $T_{\eta,-t}W^+$ is contained in at most C sets W_j.*
4. *There exist functions $\rho_1, \dots, \rho_\ell$ of class C^{r+1}, ρ_j compactly supported on W_j, such that $\sum \rho_j = 1$ on $T_{\eta,-t}(W)$, and $|\rho_j|_{C^{r+1}} \le C$.*

Proof. We do the argument for $\eta = 0$, the one for $\eta \neq 0$ being equal a part from the heavier notation. Again we must worry only about small t since for large t [2, Lemma 7.2] is perfectly all right even in our modified situation. Given $W \in \Sigma_0$ let us use d for the metric $d_{T_{-t}W}$. By Lemma 4.1 $T_{-t}W$ contains a ball in the d_W metric of size $2e^{\varsigma t}\delta$ while it has diameter bounded by $2e^{\Lambda t}\delta$ for some fixed Λ. If $x \in W$ is the center, then let $\Omega = \{z \in T_{-t}W \ : \ d(z, T_{-t}x) \le 2(e^{\Lambda t} - e^{-\Lambda t})\delta\}$. Note that if $z \in \Omega$ and $y \in \partial T_{-t}W^+ = T_{-t}\partial W^+$, then

$$d(z, y) \ge d(y, T_{-t}x) - d(z, T_{-t}x) \ge \left[e^{\varsigma t}B - 2(e^{\Lambda t} - 2^{-\Lambda t})\right]\delta \ge B\delta$$

provided that B has been chosen large enough. Moreover, by Vitali covering Lemma, for each $\alpha > 0$ there exists a set $\Gamma_\alpha := \{y_i\} \subset \Omega$ such that $d(y_i, y_j) \ge \alpha(1 - \delta_{ij})$ and for any $z \in \Omega$ there exists y_i such that $d(z, y_i) \le 5\alpha$. For each

[5] Here is the reason to introduce approximate foliations $\mathcal{E}^s, \mathcal{E}^s$. If we would have used the cone C_ρ, based on E^u, E^s, the present argument would have failed miserably. Indeed, the boundaries of two C_ρ cones at points ηt close could differ by $(\eta t)^\varpi$, a quantity that, for small t, is much larger (no matter how small is η) than the margin $\frac{\zeta}{2}t$ provided by the cone contraction under the unperturbed dynamics.

[6] Note that, contrary to [2] and similarly to [4], we do not have restricted versions of the manifolds. Such restricted version were used in [2] for clarity but they do not play any essential role.

$y \in \Gamma_{\Lambda t/5}$ we can consider a ball W_y^+ in $T_{-t}W^+$ with center y and radius $B\delta$, by the above considerations $W_y^+ \in T_{-t}W^+$. Moreover if $w \in T_{-t}W$, then there exist some $z \in \Omega$ such that $d(z,w) \leq 2e^{-\Lambda t}\delta$, hence there exists $y \in \Gamma_{\Lambda t/5}$ such that

$$d(y,w) \leq 2e^{-\Lambda t}\delta + \Lambda t \leq 2\delta.$$

In other words $\cup_{y\in\Gamma_{\Lambda t/5}} W_y \supset T_{-t}W$ while $\cup_{y\in\Gamma_{\Lambda t/5}} W_y^+ \subset T_{-t}W^+$. Moreover, the maximal number of overlaps is given by the cardinality of $\Gamma_{\Lambda t/5}$ which is uniformly bounded. Now that we have the required covering the existence of the partition of unity subordinated to it is a standard result. □

Unfortunately, we are not done yet: we need a control on higher derivatives. Thus we must restrict our set of leaves so as to insure such a control.

First of all notice that each $W \in \Sigma_0$ has a representation in a chart (U_i, Ψ_i) of the type $G_{x,F}(\xi) = x + (\xi, F(\xi))$, exactly as in [2]. Next, for each $z \in W \in \Sigma_0$, we introduce a convenient coordinate frame:[7] let $\zeta = \Psi_i(z)$, $\bar\xi \in B(0,\delta)$ such that $G_{x,F}(\bar\xi) = \zeta$, and $E = D_{\bar\xi}G_{x,F}$. We then introduce an affine change of coordinates Θ_ζ in $\mathbb{R}^d$ so that $\Theta_\zeta(0) = \zeta$, $\Theta_\zeta\left(\{(\xi,0,0)\}_{\xi\in\mathbb{R}^{d_s}}\right) = E$, $\Theta_\zeta\left(\{(0,\eta,0)\}_{\eta\in\mathbb{R}^{d_u}}\right) = D\Psi_i(\mathcal{E}^u(z))$ and $\Psi_i^{-1}\circ\Theta_\zeta(y + (0,0,\|X(z)\|_2 t)) = T_t\circ\Psi_i^{-1}\circ\Theta_\zeta(y)$ for t small. Moreover we require that $\|D_0\Theta_\zeta\Psi_i^{-1}(y)\|_2 = \|y\|$, $\|\cdot\|$ being the Euclidean norm in $\mathbb{R}^d$. Note that if $z \in U_i \cap U_j$ then, setting $\Xi_{z,i,j} = (\Psi_i^{-1}\circ\Theta_\zeta)^{-1}\circ(\Psi_j^{-1}\circ\Theta_{\zeta'})$, $\zeta' = \Psi_j(z)$, it follows

$$D_0\Xi_{z,i,j} = \begin{pmatrix} I_A & 0 & 0 \\ 0 & I_B & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where I_A, I_B are isometries. Accordingly, in the following, there is no loss of generality if we limit ourself to study the relevant objects in the same chart. Let $\overline{\Psi}_{W,z} = \Theta_\zeta^{-1}\circ\Psi_i$.

In the new coordinates the manifold W reads $\mathbb{F}_z(\xi) = (\xi, F_z(\xi))$, with $F_z(0) = D_0F_z = 0$. Let $D\mathbb{F}_z = (\mathbb{1}, S_z)$. We are now ready to state the new definition of the set Σ of admissible leaves.[8] For $M > 0$ large enough, let[9]

$$\Sigma = \big\{W \in \Sigma_0 \ :\ \forall z \in W \quad S_z(0) = 0, \\ \|D^\alpha S_z(0)\| \leq M^{|\alpha|} \ \forall\, |\alpha| \leq r+1\big\}.$$

Next, define $\phi_{W,z,t} := \overline{\Psi}_{T_{-t}W,T_{-t}z}\circ T_{-t}\circ\overline{\Psi}_{W,z}^{-1}$. Then, by construction,

$$D\phi_{W,z,t} = \begin{pmatrix} A_t & B_t & 0 \\ C_t & D_t & 0 \\ a_t & b_t & \frac{\|X(T_{-t}z)\|_2}{\|X(z)\|_2} \end{pmatrix} = \begin{pmatrix} A_t & \tilde B_t \\ \tilde C_t & \tilde D_t \end{pmatrix}.$$

Note that $D_0\phi_{W,z,t}(\xi,0,0) = (\alpha_t\cdot\xi, 0, 0)$, hence we have $C_t(0) = a_t(0) = 0$, and, by Lemma 4.1, $\|A_t^{-1}\| \leq e^{-\varsigma t}$. On the other hand $D_0\phi_{W,z,t}(\gamma\cdot\eta, \beta\cdot\eta, \tau\cdot\eta) = (0,\eta,0)$

[7] This is very similar to the coordinates used in [2, Appendix A], unfortunately we need to redo the computation because here we need a traverse foliation independent on the leaf and we have a different metric.

[8] The sophisticated reader might like to restate this definition in terms of *germs*. We prefer to keep things as elementary as possible.

[9] Here we use the usual multi-index notation from PDE: $\alpha \in \mathbb{N}^{d_s}$ is a multi-index, $|\alpha| = \sum_{i=1}^{d_s}\alpha_i$, $D^\alpha = \partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_{d_s}}^{\alpha_{d_s}}$.

were, by Lemma 3.1,

$$\|\gamma\| + \|\tau\| \leq C\varepsilon t. \tag{5.1}$$

Hence $\|B_t(0)\| + \|b_t(0)\| \leq C\varepsilon t$. Moreover, Lemma 4.1 implies

$$\|D_t(0)\| \leq e^{-\varsigma t}.$$

Let $\mathbb{F}_{z,t}$ be the map that describes $T_{-t}W$ at $T_{-t}z$ and $D\mathbb{G}_{z,t} = (\mathbb{1}, S_{z,t})$. A direct computation shows that

$$S_{z,t} = (\tilde{C}_t + \tilde{D}_t S_{z,0})(A_t + \tilde{B}_t S_{z,0})^{-1}. \tag{5.2}$$

We have finally the required correct version of [2, Lemma 7.2]

Theorem 5.2. *There exists $M, \eta_0 > 0$ such that for each $T_{\eta,-t} \in \mathcal{T}_{\eta_0}$ the set of leaves Σ is invariant (in the sense of Lemma 5.1) under each flow $T_{\eta,-t}$.*

Proof. By the usual hyperbolic theory there exists $\tau > 0$ such that, for M large enough the statement is true for all $t \geq \tau$. The problem is then limited to small times. Given Lemma 5.1 the Theorem is proven once we show that the bound on the higher derivatives of the leaves are preserved under the dynamics.

Now, let $z \in W \in \Sigma$, $W_{\eta,t} = T_{\eta,-t}W$ and $\mathbb{F}_{z,\eta,t}$ be the function that describes $W_{\eta,t}$ at the point $T_{\eta,-t}z$, let $D\mathbb{F}_{z,\eta,t} = (\mathbb{1}, S_{z,\eta,t})$. Note that $\mathbb{F}_{z,t} = \mathbb{F}_{z,0,t}$. In analogy with what we have done before we define $\phi_{W,z,\eta,t} := \overline{\Psi}_{T_{\eta,-t}W,T_{\eta,-t}z} \circ T_{\eta,-t} \circ \overline{\Psi}^{-1}_{W,z}$, then[10]

$$\|D\phi_{W,z,\eta,t} - D\phi_{W,z,t}\|_{\mathcal{C}^r} \leq C|\eta|t\,.$$

It follows that, for each $r' \leq r$,

$$\|S_{z,\eta,t} - S_{z,t}\|_{\mathcal{C}^{r'}} \leq C|\eta|tM^{r'}. \tag{5.3}$$

Remark that, by construction, $S_{z,\eta,t}(0) = 0$. To continue note that (5.2) can be written as

$$\begin{aligned} S_{z,\eta,t}(\xi) &= \tilde{D}_t(0)S_{z,\eta,0}(\xi)A_t(0)^{-1} + \sum_{k=0}^{\infty} \Xi_k(\mathbb{F}_{z,\eta,t}(\xi), S_{z,\eta,0}(\xi)) \\ \Xi_k(x,S) &= (-1)^k \Omega(x,S)[\tilde{B}_t(x)SA_t(x)^{-1}]^k \\ \Omega(x,S) &= (\tilde{C}_t(x) + [\tilde{D}_t(x) - \tilde{D}_t(0)]S)A_t(x)^{-1} \\ &\quad + \tilde{D}_t(0)SA_t(0)^{-1}[\mathbb{1} - A_t(0)A_t(x)^{-1} + \tilde{B}_t(x)SA_t(x)^{-1}]. \end{aligned} \tag{5.4}$$

The key point of the above representation being that $\Xi_k(x,S) = \widetilde{\Xi}_k(x,S,\dots,S)$ where $\widetilde{\Xi}_k(x,S_1,\dots,S_{k+2})$ is linear in each of the S_i variables. Moreover, remembering (4.1) and the discussion around (5.1),

$$\begin{aligned} \|\widetilde{\Xi}_k(0,S_1,\dots,S_{k+2})\| &\leq C\varepsilon t \prod_{i=1}^{k+2} \|S_i\| \\ \|\widetilde{\Xi}_k(\cdot,S_1,\dots,S_{k+2})\|_{\mathcal{C}^r} &\leq Ct \prod_{i=1}^{k+2} \|S_i\| \end{aligned} \tag{5.5}$$

[10] The estimate follows since $\overline{\Psi}_{T_{\eta,-t}W,T_{\eta,-t}z} \circ \overline{\Psi}^{-1}_{T_{-t}W,T_{-t}z}$ is an affine change of coordinates that differs from the identity by less than $K|\eta|t$. Here is the other key point in which we use the smoothness of the foliation $\mathcal{E}^u$, if we would have used the foliation E^u now we would have a bound of the type $(|\eta|t)^{\varpi}$, useless for our purposes.

where the $\mathcal{C}^r$ norm is taken in a sufficiently small neighborhood of zero.

Let us look at the case $\eta = 0$ first. By differentiating (5.4), we have

$$\begin{aligned} D^\alpha S_{z,t}(\xi) =& D_t(0) D^\alpha S_{z,0}(\xi) A_t(0)^{-1} \\ &+ \sum_{k=0}^{\infty} \sum_{|\beta_1|+\cdots+|\beta_{k+2}|=|\alpha|} \widetilde{\Xi}_k(\mathbb{F}_{z,t}(\xi), D^{\beta_1} S_{z,0}(\xi), \dots, D^{\beta_{k+2}} S_{z,0}(\xi)) \\ &+ \sum_{k=0}^{\infty} \sum_{\substack{n\le|\alpha| \\ |\beta_1|+\cdots+|\beta_n|<|\alpha| \\ |\beta_i|>0}} \Omega_{k,\beta_1,\cdots,\beta_n}(\xi, S_{z,0}(\xi), D^{\beta_1} S_{z,0}(\xi) D^{\beta_n} S_{z,0}(\xi)) \end{aligned}$$

for certain functions $\Omega_{k,\beta_1,\cdots,\beta_n}$ that, by the second of the (5.5) satisfy the bound

$$\begin{aligned} &\Omega_{k,\beta_1,\cdots,\beta_n}(0,0,S_1,\dots,S_n) = 0 \quad \forall \sum_{i=1}^n |\beta_i| < k+2 \\ &\|\Omega_{k,\beta_1,\cdots,\beta_n}(0,0,S_1,\dots,S_n)\| \le Ct \prod_{i=1}^n \|S_i\|. \end{aligned}$$

Thus, remembering the first of (5.5), there exists $K > 0$ such that

$$\|D^\alpha S_{z,t}(0)\| \le e^{-\varsigma t}\|D^\alpha S_{z,0}(0)\| + K\varepsilon t M^{|\alpha|} + KtM^{|\alpha|-1}$$

Thus Σ is invariant for T_{-t} provided M is large enough. Analogously, for $\eta \neq 0$,

$$\|D^\alpha S_{z,\eta,t}(0)\| \le e^{-\varsigma t}\|D^\alpha S_{z,\eta,t}(0)\| + Kt(\varepsilon + |\eta| + M^{-1})M^{|\alpha|},$$

which implies the Lemma provided η_0 is small enough. □

6. New Banach spaces and old proofs

Let us call $\tilde{\mathcal{B}}^{p,q}$ the Banach spaces defined in [2] (there called simply $\mathcal{B}^{p,q}$). We define new norms exactly as before [2, (2.3)], but with the set Σ of admissible leaves as defined above. The new Banach spaces, that we call $\mathcal{B}^{p,q}$, are then defined again as the closure of the smooth functions in such norms.

As already noted, the set Σ is contained in the old one (possibly with different parameters), moreover now the conclusions of [2, Lemma 7.2] hold. This means that the proofs of [2, Lemmata 4.1, 4.2, 4.3] hold verbatim. Indeed the only problem in the published proofs was the lack of invariance of Σ for small times. The results of [2, Section 5] are correct provided [2, Section 4] is correct. Since Theorem 5.2 holds true for an open set of dynamics, the perturbations results of [2, Section 6] hold true as well.

The only problem left is then with [2, Lemma 4.4], whose proof is based on [2, Lemma 2.2] (and in turn on [4, Lemma 2.1]). The only use of [2, Lemma 4.4] in [2] is to prove [2, Lemma 4.5]. Unfortunately, even though the space is only slightly different form the one in [4], as far as we see [4, Lemma 2.1] could be false in the present context. Indeed, the proof in [4] uses in a crucial way that in each chart the cone field is constant, a property that we no longer have. To overcome this obstacle we bypass [2, Lemma 4.4] and provide a direct proof of [2, Lemma 4.5] following the strategy in [6].

7. Quasi-compactness of the resolvent

The following Lemma immediately implies [2, Lemma 4.5].

Lemma 7.1. *For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $q + p < r$, and $z \in \mathbb{C}$, $\Re(z) = a > 0$ the operator $R(z) : \mathcal{B}^{p,q} \to \mathcal{B}^{p,q}$ has spectral radius bounded by a^{-1} and essential spectral radius bounded by $(a + \bar{p}\lambda)^{-1}$.*

Proof. Since the conclusions of [2, Lemmata 4.1, 4.3] hold, the estimate of the spectral radius is as before. It remains to prove the bound on the essential spectral radius. First of all notice that there exists $K > 0$ such that $\mathcal{L}_t \in L(\mathcal{B}^{p,q}, \tilde{\mathcal{B}}^{p,q})$ and $\mathcal{L}_t \in L(\tilde{\mathcal{B}}^{p-1,q+1}, \mathcal{B}^{p-1,q+1})$ for all $t \geq K$.[11] From this it follows that the operators

$$\begin{aligned} R_{K,m}(z) &= \frac{1}{(m-1)!} \int_{3K}^{\infty} t^{m-1} e^{-zt} \mathcal{L}_t dt \\ &= \mathcal{L}_K \left[\frac{e^{-2Kz}}{(m-1)!} \int_K^{\infty} (t+2K)^{m-1} e^{-zt} \mathcal{L}_t dt \right] \mathcal{L}_K \end{aligned}$$

are compact, as operators in $L(\mathcal{B}^{p,q}, \mathcal{B}^{p-1,q+1})$. Indeed, the incorrect proof of the compactness of $R(z)$ in [2] holds correct for the operator in square brackets, since no small times are involved, yielding compactness as an operator in $L(\tilde{\mathcal{B}}^{p,q}, \tilde{\mathcal{B}}^{p-1,q+1})$. The result then follows by the above continuity properties of $\mathcal{L}_K$. Thus, setting

$$Q_{K,m}(z) = \frac{1}{(m-1)!} \int_0^{3K} t^{m-1} e^{-zt} \mathcal{L}_t dt,$$

we have $R(z)^m = R_{K,m} + Q_{K,m}$, and [2, Lemma 4.1] implies

$$\|Q_{K,m}(z)\|_{p,q} \leq C_{p,q} \frac{K^m}{m!}.$$

We can then conclude by the usual Hennion's argument [5] based on Nussbaum's formula [8]. Let us recall the argument. Let $B = \{h \in \mathcal{B}^{p,q} \ : \ \|h\|_{p,q} \leq 1\}$ and $B_m = R_{K,m}(z)B$. By the above discussion B_m is compact in $\mathcal{B}^{p-1,q+1}$. Thus, for each $\epsilon > 0$ there are $h_1, \ldots, h_{N_\epsilon} \in B_m$ such that $B_m \subseteq \bigcup_{i=1}^{N_\epsilon} U_\epsilon(h_i)$, where $U_\epsilon(h_i) = \{h \in \mathcal{B} \mid \|h - h_i\|_{p-1,q+1} < \epsilon\}$. For $h \in B_m \cap U_\epsilon(h_i)$, [2, Lemma 4.3] implies

$$\begin{aligned} &\|R(z)^n (h - h_i)\|_{p,q} \leq \|R(z)^{n-m} R_{K,m}(h - h_i)\|_{p,q} + C_{p,q} a^{-n+m} \frac{K^m}{m!} \|h - h_i\|_{p,q} \\ &= C_{p,q,\lambda'} (a + \bar{p}\lambda')^{-n+m} a^{-m} \ + C_{p,q,\lambda',a_0} |z| a^{-n+m} \epsilon + C_{p,q} a^{-n+m} \frac{K^m}{m!}. \end{aligned}$$

Choosing $\epsilon = a^n (a + \lambda\bar{p})^{-n+1}$ and $m = \delta n$, for δ small enough, we conclude that, for each $\lambda'' \in (0, \lambda)$, for each $n \in \mathbb{N}$ the set $R(z)^n B$ can be covered by a finite number of $\|\cdot\|_{p,q}$–balls of radius $C(a + \bar{p}\lambda'')^{-n}$, which implies that the essential spectral radius of $R(z)$ cannot exceed $(a + \bar{p}\lambda)^{-1}$. □

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[11] This follows since the set of leaves in [2] converge to the stable leaves under the dynamics, hence after some time they will belong to Σ.

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